Discontinuities

A real–valued function f of one real variable has discontinuity at x_0 iff

- x_0 is the cluster point of D_f and $x_0 \notin D_f$ (we can try to calculate the limit of f at x_0 but f is not defined at x_0)

  or

- f is discontinuous at x_0, i.e. $x_0 \in D_f$ but the limit of f at x_0 does not exist or is different than $f(x_0)$.

Exercise 1. Discuss discontinuities of function f

a) $f(x) = \dfrac{\sin(x-2)}{x^2-3x+2}$

We have $f(x) = \dfrac{\sin(x-2)}{(x-2)(x-1)}$ so $D_f = \mathbb{R} \setminus \{1,2\}$. Function f is continuous on its domain[1] (in the top we have the composition of two continuous functions, in the bottom — polynomial).

Points 1 and 2 do not belong to domain but they are cluster points of domain so f has discontinuities at 1 and at 2. In order to determine their types we have to calculate limits of f at these points.

We have

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\sin(x-2)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{\sin(x-2)}{x-2} \cdot \frac{1}{x-1} = 1 \cdot 1 = 1$$

because if $x \to 2$ then $x-2 \to 0$ and $x-1 \to 1$. Hence f has removable discontinuity (I type) at 2.

For the limit at 1 we have to calculate one–sided limits because there is not an indeterminate form (top tends to some number different than zero, bottom to zero). We obtain

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\sin(x-2)}{(x-2)(x-1)} = \left[\frac{\sin(-1)}{-1\cdot 0^+}\right] = +\infty$$

because $\sin(-1) = -\sin 1 < 0$ (look at the graph of sine function) and if $x \to 1+$ (which means that x is close to 1 and greater than 1) then $x-1$ tends to 0 (but is positive).

Analogously ($x \to 1+$ means that x is close to 1 and less than 1)

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{\sin(x-2)}{(x-2)(x-1)} = \left[\frac{\sin(-1)}{-1\cdot 0^-}\right] = -\infty.$$

Therefore f has infinite jump (the discontinuity of II type) at 1.

b) $f(x) = \operatorname{arccot} \dfrac{x-1}{3-x}$

We see that $D_f = \mathbb{R} \setminus \{3\}$. Function f is continuous (as the composition of two continuous functions: arccot and rational function). Hence f has the discontinuity at 3.

If $x \to 3^+$ (x is close to 3 but greater than 3), then:

- $x - 1 \to 2$
- $3 - x \to 0^-$
- $\frac{x-1}{3-x} \to -\infty$ because we have $[\frac{2}{0^-}]$
- $\operatorname{arccot} \frac{x-1}{3-x} \to \pi$

so $\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} \operatorname{arccot} \dfrac{x-1}{3-x} = \pi.$

[1] We say that it is just *continuous*.

If $x \to 3^-$ (x is close to 3 but less than 3), then:

- $x - 1 \to 2$
- $3 - x \to 0^+$
- $\frac{x-1}{3-x} \to +\infty$ because we have $[\frac{2}{0^+}]$
- $\operatorname{arccot} \frac{x-1}{3-x} \to 0$

so $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} \operatorname{arccot} \dfrac{x-1}{3-x} = 0.$

Therefore f has I type discontinuity (finite jump) at 3.

c) $f(x) = \begin{cases} (1-x)^{x^2} & \text{if} \quad x < 0 \\ \arcsin x & \text{if} \quad x \in \langle 0, 1\rangle \\ \ln(x-1) & \text{if} \quad x > 1 \end{cases}$

We see that $D_f = \mathbb{R}$. Function f is continuous on intervals: $(-\infty, 0)$, $\langle 0, 1\rangle$, $(1, +\infty)$. It may be discontinuous at 0 and 1, we have to check it.

Since

$$f(0) = (1-0)^0 = 1^0 = 1,$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1-x)^{x^2} = 1,$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \arcsin x = \arcsin 0 = 0,$$

f is continuous at 0.

Since

$$f(1) = \arcsin 1 = \frac{\pi}{2},$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \arcsin x = \frac{\pi}{2},$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \ln(x-1) = \left[\begin{array}{c} u = x - 1 \\ x \to 1^+ \Rightarrow u \to 0^+ \end{array} \right] = \lim_{u\to 0^+} \ln(u) = -\infty,$$

f has infinite jump at 1.

d) $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if} \quad x \in \langle -1, 0) \\ 2 & \text{if} \quad x = 0 \\ -3 & \text{if} \quad x = 5 \end{cases}$

We see that $D_f = \langle -1, 0\rangle \cup \{5\}$. Function f is continuous on interval $\langle -1, 0)$ and it is continuous at 5 (isolated point of D_f; remember that at isolated points of domain functions are always continuous — it follows directly from the definition of continuity). Function f may be discontinuous at 0, we have to check it.

We obtain

$$f(0) = 2,$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x}{x} = 1,$$

so f has removable discontinuity at 0 (we can redefine the function by the change of its value at 0 to obtain new function which is continuous at 0).

Limits of functions of two variables

I Sometimes it is easy, you have to remember that if we calculate limit $\lim\limits_{(x,y)\to(a,b)} f(x,y)$, then x tends to a and b tends to y simultaneously. If you obtain an indeterminate form, then try to simplify or rewrite the function in 'better' form and apply formulas for limits of functions of one variable, for example:

1. $\lim\limits_{(x,y)\to(2,4)} \dfrac{x+y}{x^2-y^2} = \dfrac{2+4}{4-16} = \dfrac{1}{2};$

2. $\lim\limits_{(x,y)\to(2,-2)} \dfrac{x+y}{x^2-y^2} \overset{\left[\frac{0}{0}\right]}{=} \lim\limits_{(x,y)\to(2,-2)} \dfrac{x+y}{(x-y)(x+y)} = \lim\limits_{(x,y)\to(2,-2)} \dfrac{1}{x-y} = \dfrac{1}{4};$

3. $\lim\limits_{(x,y)\to(2,0)} (1+3xy)^{\frac{x-1}{y}} \overset{[1^\infty]}{=} \lim\limits_{(x,y)\to(2,0)} \left[(1+3xy)^{\frac{1}{3xy}}\right]^{3x(x-1)} = e^6;$

4. $\lim\limits_{(x,y)\to(2,1)} (1+3xy)^{\frac{x-1}{y}} = 7^1 = 7.$

II If you suspect that the limit does not exists, apply the Heine definition:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \iff$$

$$\iff \forall_{\{(x_n,y_n)\}} \left[(x_n,y_n)\in D_f, (x_n,y_n)\neq(a,b), \lim_{n\to+\infty}(x_n,y_n)=(a,b) \Rightarrow \lim_{n\to+\infty} f(x_n,y_n) = L\right].$$

- If we can find one sequence of arguments, convergent to (a,b), satisfying condition $(x_n,y_n)\neq(a,b)$, such that $\lim\limits_{n\to+\infty} f(x_n,y_n)$ does not exist, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.
- If we can find two sequences of arguments $\{(x'_n,y'_n)\}$ and $\{(x''_n,y''_n)\}$, both convergent to (a,b), satisfying conditions $(x'_n,y'_n)\neq(a,b)$ and $(x''_n,y''_n)\neq(a,b)$, such that $\lim\limits_{n\to+\infty} f(x'_n,y'_n) \neq \lim\limits_{n\to+\infty} f(x''_n,y''_n)$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 5. Let us try to calculate $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$.

Let $f(x,y) = \dfrac{xy}{x^2+y^2}$. Then $D_f = \mathbb{R}^2 \setminus \{(0,0)\}$.
Let the first sequence has general term $(x'_n,y'_n) = \left(\frac{1}{n},\frac{1}{n}\right)$. Then

$$\lim_{n\to+\infty} f(x'_n,y'_n) = \lim_{n\to+\infty} f\left(\frac{1}{n},\frac{1}{n}\right) = \lim_{n\to+\infty} \frac{\frac{1}{n}\cdot\frac{1}{n}}{\frac{1}{n^2}+\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \lim_{n\to+\infty} \frac{1}{2} = \frac{1}{2}.$$

Let the second sequence has general term $(x''_n,y''_n) = \left(0,\frac{1}{n}\right)$. Then

$$\lim_{n\to+\infty} f(x''_n,y''_n) = \lim_{n\to+\infty} f\left(0,\frac{1}{n}\right) = \lim_{n\to+\infty} \frac{0\cdot\frac{1}{n}}{0+\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{0}{\frac{1}{n^2}} = \lim_{n\to+\infty} 0 = 0.$$

Obviously $\frac{1}{2}\neq 0$ so $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

Example 6. Let us try to calculate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$.

Let $f(x,y) = \dfrac{x^2y}{x^4+y^2}$. Then $D_f = \mathbb{R}^2 \setminus \{(0,0)\}$.

First, we consider infinitely many sequences with general term $(x'_n, y'_n) = \left(\frac{1}{n}, \frac{k}{n}\right)$, where k is any real number. Then

$$\lim_{n\to+\infty} f(x'_n, y'_n) = \lim_{n\to+\infty} f\left(\frac{1}{n}, \frac{k}{n}\right) = \lim_{n\to+\infty} \frac{\frac{1}{n^2}\cdot\frac{k}{n}}{\frac{1}{n^4}+\frac{k}{n^2}} = \lim_{n\to+\infty} \frac{\frac{k}{n}}{\frac{1}{n^2}+k} = 0.$$

It means that **if** this limit exists, then it is equal to 0. But the word 'if' is important. Let $(x''_n, y''_n) = \left(\frac{1}{n}, \frac{1}{n^2}\right)$. Then

$$\lim_{n\to+\infty} f(x''_n, y''_n) = \lim_{n\to+\infty} f\left(\frac{1}{n}, \frac{1}{n^2}\right) = \lim_{n\to+\infty} \frac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^4}+\frac{1}{n^4}} = \lim_{n\to+\infty} \frac{\frac{1}{n^4}}{\frac{2}{n^4}} = \lim_{n\to+\infty} \frac{1}{2} = \frac{1}{2}.$$

Now we are sure that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ does not exist.

III Sometimes the Squeeze Theorem may be useful.

Example 7. Calculate $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2}$.

For any $(x, y) \neq (0, 0)$ we have

$$0 \leq \frac{x^2y^2}{x^2+y^2} \leq \frac{x^2y^2}{x^2} = y^2.$$

Since $\lim\limits_{(x,y)\to(0,0)} 0 = \lim\limits_{(x,y)\to(0,0)} y^2 = 0$, we obtain $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} = 0$.

Remarks:

- the sequence $\{(a+\frac{1}{n}, b+\frac{k}{n})\}$ tends to (a, b);
- if you obtain the same limit of $f(a+\frac{1}{n}, b+\frac{k}{n})$ for any k, try other sequences, for example $\{(a+\frac{1}{n^2}, b+\frac{k}{n})\}$ or $\{(a+\frac{1}{n}, b+\frac{k}{n^2})\}$;
- if you suspect that $\lim_{(x,y)\to(a,b)} f(x, y) = L$, you may try to show that $\lim_{(x,y)\to(a,b)} |f(x, y)-L| = 0$ using the Squeeze Theorem;
- for limits at $(0, 0)$ the polar coordinates $x = r\cos t$, $y = r\sin t$ may be better (you know them from algebra).