MACRO, sem. 1

Examples of tasks for learning outcomes

Note: Only the definitions, theorems, formulas given in the lectures can be used.

E1 — domain

Find and sketch the domain of

a) $z = \arcsin\frac{x+2y}{4} + \ln(x - \sqrt{y+1})$

b) $z = \arcsin(y+x^2) - \arccos(y-x^2) + \ln(2x - x^2)$

c) $z = \frac{\arcsin(y-x^2)}{\arccos(y+x^2)} + \sqrt{x^2+3x}$

d) $z = \frac{\arcsin(y-x^2)}{\arccos(y+x^2)} + \arctan(x^2 + \frac{3}{y} - 1)$

e) $z = \frac{e^{x+y} - \ln(xy)}{\arccos(x^2+y^2-2)} + \sqrt{1-x}$

f) $z = \arcsin(x+y-1) + \sqrt{-y}\cdot\arctan(x + \frac{y}{x})$

g) $z = \arccos(1-x^2-y^2) + \sqrt{y-|x|}\cdot\ln y$

h) $z = \sqrt{y}\,\ln(y-|x|) - \arcsin(4-x^2-y^2)$

i) $z = \arctan\sqrt{x^2+y^2-4} + \log_2(x-y)$

j) $z = \ln\left(\arccos\frac{x}{2}\right) + \sqrt[4]{e^x - y}$

k) $z = \sqrt{\arccos\frac{x}{2}} + \ln(e^y - x)$

l) $z = \ln\left(\arccos\frac{y}{2}\right)\cdot\sqrt{y-x^2-1}$

m) $z = \frac{\ln(1-x^2) + \sqrt{2-x^2-y^2}}{\pi - 4\arctan(x+y-1)}$

n) $z = \frac{\sqrt{1-\frac{1}{x}}}{\arcsin(x^2+y^2)}$

o) $z = \sqrt[4]{\frac{1}{x} - 1} + 2\arccos\frac{x^2-y}{2}$

p) $z = \arccos(x-y) + \sqrt{\frac{2}{x} - 3}$

q) $z = 3\arcsin(x-y) + x\ln(|y+1|-2)$

r) $z = \sqrt{\pi - 3\arccos\frac{x+y}{2}} + \ln(1+y-x^2)$

s) $z = \sqrt{\pi - 3\arcsin\frac{x+y}{\sqrt{3}}} + \ln(1-y^2-x^2)$

t) $z = \sqrt{\pi + 3\arccos\frac{x+y}{2}}\cdot\ln\left(\pi + 3\arctan\frac{x+y}{\sqrt{3}}\right)$

u) $z = \ln\left(\arcsin\frac{x-|y|+1}{2} - \frac{\pi}{6}\right) + e^y\sqrt{\operatorname{arccot}(x-2) - \frac{3}{4}\pi}$

E2 — inverse function

Find inverse function of

a) $f\colon\ y = 2\arcsin(3x-1)$

b) $f\colon\ y = 1 - \sqrt{2\pi\arcsin\frac{x}{3}}$

c) $f\colon\ y = \ln\left(3\arccos\frac{x-1}{2} - 2\pi\right)$

d) $f\colon\ y = 3e^{2\arctan\sqrt{x}}$

e) $f\colon\ y = 1 - e^{\sqrt[4]{\arccos x}}$

f) $f\colon\ y = 1 + e^{2\arctan\sqrt{x}},\ x \geq 1$

g) $f\colon\ y = 2e^{\arcsin\frac{1}{x}}$

h) $f\colon\ y = e^{-2\arccos\frac{1}{x}}$

i) $f\colon\ y = 1 + 2\ln(\arctan x + \pi)$

j) $f\colon\ y = \ln^3\left(\frac{x}{1-x}\right)$

k) $f\colon\ y = \sqrt[4]{3\pi - 4\operatorname{arccot}\frac{x}{2}}$

l) $f\colon\ y = \pi - 2\arcsin(\ln x)$

m) $f\colon\ y=\pi+\arccos(\ln x)$

n) $f\colon\ y=\ln(4\operatorname{arccot}\sqrt{x})$

o) $f\colon\ y=\arccos(-\sqrt{x})-1$

p) $f\colon\ y=\cos^2 x,\ x\in(-\pi,-\frac{\pi}{2}\rangle$

q) $f\colon\ y=\left(3\pi+2\arcsin\frac{x}{2}\right)^3$

r) $f\colon\ y=1+\operatorname{arccot}\frac{3}{x-1}$

s) $f\colon\ y=\left(\arctan\frac{3}{x+1}\right)^5$

t) $f\colon\ y=\sqrt{\pi-4\arctan\frac{x-1}{3}}$

u) $f\colon\ y=2\arcsin\frac{1}{\ln x-1}$

E3 — properties of functions

- Calculate the exact value of

a) $\tan\frac{\pi}{4}+\sin(\arctan(-3))$

b) $\tan(\arccos\frac{2}{3})-\sin\pi$

c) $\cos(3\pi)+\cos(2\arccos\left(-\frac{1}{5}\right))$

d) $\arcsin(\sin 6)+3\arccos(-\frac{1}{2})$

- Sketch the graph of given function

a) $y=-\left|\ln|x|\right|$

b) $y=\arctan|x-1|+\pi$

c) $y=\left|\frac{\pi}{2}-\arccos\left(x+\frac{1}{2}\right)\right|$

d) $y=e^{|x|-1}$

- Is the function one–to–one? Apply appropriate definition.

a) $f(x)=e^x+\cosh x$

b) $f(x)=e^x+\sinh x$

c) $f(x)=\frac{x}{x^2-4}$

d) $f(x)=\frac{2x-1}{3-x}$

e) $f(x)=\frac{x^2}{x-1}$

f) $f(x)=x^3-3x$

E4 — the limit of sequence

- Calculate the limit of sequence with general term a_n, where

a) $a_n=\left(\frac{2n+1}{2n-3}\right)^{3n}$

b) $a_n=\left(\frac{4n+1}{4n+3}\right)^{\frac{3n+1}{2}}$

c) $a_n=\left(\frac{n^2+2}{n^2+n}\right)^{3n-1}$

d) $a_n=\left(\frac{n^2-3n}{n^2+1}\right)^{\frac{1}{2n}}$

e) $a_n=\left(\frac{n^2+2n}{n^2+3}\right)^{-\frac{n}{2}}$

f) $a_n=\sqrt[n]{3^n+2\cos n}$

g) $a_n=\sqrt[n]{2n+\sin\frac{1}{n}}$

h) $a_n=\frac{1}{n}\left(\sqrt{n^2+1}-\sqrt{n^2+3n-2}\right)$

i) $a_n=\frac{\sqrt{2n+1}-\sqrt{2n-2}}{\sqrt{4n+2}-\sqrt{4n-3}}$

j) $a_n=\sqrt[n]{\left(\frac{2n-1}{2n+3}\right)^{3n^2+n}}$

k) $a_n=n(|3n+i|-|2-3ni|)$

l) $a_n=|n+2+i|-|1+ni|$

m) $a_n=|2n+1+3i|-|1-2in|$

n) $a_n=\frac{1}{n+1}\left(|3n+5i|-|1-in|\right)$

o) $a_n=\frac{1}{n}(|2n+ni|-|5-ni|)$

p) $a_n=\frac{\sqrt{n+1}-\sqrt{n+3}}{\sqrt{2n+3}-\sqrt{2n+1}}$

q) $a_n=\frac{\sqrt{n^2+2n}-\sqrt{n^2+3n-1}}{\left(1+\frac{1}{n}\right)^n}$

r) $a_n=\frac{\sqrt[n]{n}-3\sqrt[n]{2}}{\sqrt{4n^2+1}-\sqrt{4n^2+n}}$

E5 — the limit of function

- Calculate

a) $\lim\limits_{x\to-\infty}\left(\frac{2+3x}{7+3x}\right)^{\frac{x}{2}}$

b) $\lim\limits_{x\to-\infty}\left(\frac{2+3x}{7+3x}\right)^{\frac{2}{x}}$

c) $\lim\limits_{x\to 1}(3x-2)^{\frac{1}{x^2-1}}$

d) $\lim\limits_{x\to 0}\frac{\arcsin 2x}{\sqrt{x+4}-2}$

e) $\lim\limits_{x\to 0}\frac{2^x-3^x}{\tan\frac{x}{2}}$

f) $\lim\limits_{(x,y)\to(2,0)}\frac{\sin xy}{xy+3y}$

g) $\lim\limits_{(x,y)\to(0,1)}\frac{xy}{x^2+y^2-1}$

h) $\lim\limits_{(x,y)\to(0,0)}\frac{xy^2}{x^2+y^2}$

i) $\lim\limits_{(x,y)\to(0,0)}(1+2x^2+2y^2)^{\frac{x+3}{x^2+y^2}}$

- Determine types of discontinuity of function f. Where is the function continuous?

a) $f(x)=\frac{1}{x}\operatorname{arccot}\frac{3}{x-1}$

b) $f(x)=xe^{\frac{1}{x}}$

c) $f(x)=\frac{\sin(x-1)}{x^2-1}$

d) $f(x)=\frac{1-\cos^2 x}{x^2+2x}$

- For which a function f is continuous?

a) $f(x)=\begin{cases}\frac{3^x-2^x}{x} & \text{if } x>0\\ \ln(a^2+2x) & \text{if } x\le 0\end{cases}$

b) $f(x)=\begin{cases}\frac{x^2-1}{2-\sqrt{3x+1}} & \text{if } x<1\\ a\sin x & \text{if } x\ge 1\end{cases}$

c) $f(x)=\begin{cases}\frac{\tan 2x}{x^2+4x} & \text{if } x>0\\ \arcsin(2a-3x) & \text{if } x\le 0\end{cases}$

d) $f(x)=\begin{cases}\frac{\tan 2x}{x^2+4x} & \text{if } x>-4\\ \arcsin(2a-3x) & \text{if } x\le -4\end{cases}$

e) $f(x)=\begin{cases}\frac{\tan(x^2+4x)}{x} & \text{if } x>0\\ \arcsin(2a-3x) & \text{if } x\le 0\end{cases}$

f) $f(x)=\begin{cases}\arcsin x & \text{if } x\in\langle-1,1\rangle\\ \frac{1}{a} & \text{if } x=2\end{cases}$

- Calculate $f'(x_0)$ directly from the definition

a) $f(x)=\sqrt{x^2+2x},\ x_0=1$

b) $f(x)=\ln(2x+3),\ x_0=2$

c) $f(x)=\ln(2x-3),\ x_0=2$

d) $f(x)=e^{3x-1},\ x_0=0$

e) $f(x)=\cos^2(2x),\ x_0=0$

f) $f(x)=\frac{3}{2x+1},\ x_0=-3$

g) $f(x)=\frac{1}{x^2-2x},\ x_0=3$

h) $f(x)=3\arcsin(x+1),\ x_0=-1$

i) $f(x)=\frac{2x-3}{x+3},\ x_0=3$

j) $f(x)=\sqrt{16-3x^2},\ x_0=2$

k) $f(x)=\ln(1-3x),\ x_0=-1$

l) $f(x)=2^{1-x},\ x_0=1$

m) $f(x)=\cos(x+3),\ x_0=-3$

n) $f(x)=\arcsin\frac{x-2}{3},\ x_0=2$

o) $f(x)=\frac{x^2}{x+1},\ x_0=1$

E6 — monotonicity and extreme values

Test the monotonicity and find extreme values of

a) $f(x)=(x^2-3)e^{-x^2}$

b) $f(x)=\ln\frac{e^x}{x-1}$

c) $f(x)=x^3e^{\frac{1}{x}}$

d) $f(x)=x^3e^{-\frac{1}{x}}$

e) $f(x)=xe^{\frac{1}{x^2}}$

f) $f(x)=\frac{1}{x}e^{\frac{1}{2}x^2}$

g) $f(x) = \dfrac{e^{3x}}{6x-1}$

h) $f(x) = \dfrac{x^2-x+1}{x-1}$

i) $f(x) = (2x^2+2x-7)e^{-2x}$

j) $f(x) = \arctan\dfrac{1}{x} + \ln(x^2+1)$

k) $f(x) = \ln(x+1) + x^2 - x$

l) $f(x) = \ln(x^3-3x)$

m) $f(x) = \dfrac{1}{x}e^{-x}$

n) $f(x) = \dfrac{1}{x}e^{2x}$

o) $f(x) = \dfrac{x}{e^{x^2}}$

p) $f(x) = 4\ln x + x^2 - 6x$

q) $f(x) = x^2 - 4\ln x^2$

r) $f(x) = \sqrt{\dfrac{x}{x^2+1}}$

s) $f(x) = \arcsin x + 2\sqrt{1-x^2}$

t) $f(x) = (x^2+4x+2)e^{-2x}$

u) $f(x) = x^2+4x+6\ln(2-x)$

v) $f(x) = \dfrac{x^2+x-3}{x+1} - 4\ln(x+1)$

E7 — application of derivatives

- Find the equation of line which passes through $A(0,2)$ and is tangent to $y = xe^{-x^2}$.
- Find the equation of line which passes through $A(0,-1)$ and is tangent to $y = x\ln x$.
- Find tangent and normal lines to given curve at the point corresponding to t_0

a) $\begin{cases} x = t\cos(t^2-1) \\ y = 3e^{2t-2} \end{cases}$ $t_0 = 1$

b) $\begin{cases} x = t^3-2t+2 \\ y = \ln(t^2+1) \end{cases}$ $t_0 = 0$

c) $\begin{cases} x = \arctan\frac{1}{t} \\ y = t\sin(t+1) \end{cases}$ $t_0 = -1$

d) $\begin{cases} x = \frac{t^2}{2t+5} \\ y = t\ln(2t+5) \end{cases}$ $t_0 = -2$

- Find points at which tangent line to given curve is perpendicular to line l

a) $y = \frac{x-1}{2x+3}$, $l:\ x+5y-1=0$

b) $\begin{cases} x = t\ln t \\ y = 2t^2\ln t + t^2 \end{cases}$, $l:\ x+4y+2=0$

c) $\begin{cases} x = t - \arctan\frac{1}{t} \\ y = t+\ln(t^2+1) \end{cases}$, $l:\ 2x+y+2=0$

d) $y = xe^{\frac{1}{x}}$, $l:\ x=3$

e) $\begin{cases} x = e^{-t}(t+1) \\ y = t^2e^{-t} \end{cases}$, $l:\ x-3y+5=0$

f) $\begin{cases} x = \ln\frac{e^t}{1-t} \\ y = 2t^3-9t^2+12t+1 \end{cases}$, $l:\ x+6y+5=0$

- Find points at which normal line to given curve is perpendicular to line l

a) $\begin{cases} x = e^{-t}(t+1) \\ y = t^2e^{-t} \end{cases}$, $l:\ 2x-y+5=0$

b) $\begin{cases} x = \arccos t - \sqrt{1-t^2} \\ y = 2t^3-3t^2+\frac{1}{\sqrt{2}} \end{cases}$, $l:\ 3x+y=0$

c) $\begin{cases} x = 3\arcsin\sqrt{t} \\ y = 2\sqrt{t-t^2} \end{cases}$, $l:\ 4x-3y=1$

d) $\begin{cases} x = 3\arcsin\sqrt{t} \\ y = 2\sqrt{t-t^2} \end{cases}$, $l:\ 2x+y=1$

e) $\begin{cases} x = t\ln t \\ y = t(1+\ln^2 t) \end{cases}$, $l:\ 2x-y=1$

f) $\begin{cases} x = t^2+2t+3 \\ y = \arcsin t - \sqrt{1-t^2}) \end{cases}$, $l:\ x=2y$

g) $\begin{cases} x = \ln(t^2+1) \\ y = t-\arctan t \end{cases}$, $l:\ x+2y=3$

h) $y = \ln(1+\tan^2 x)$, $l:\ y = 2x\sqrt{3}$

- Find normal line to $x = \ln(1-2t)$, $y = \operatorname{arccot} t$ which is parallel to $2x+y-3=0$.

- Use the differential to find the approximate value of the expression (the result should be written in the decimal form)

a) $\sqrt{(9.001)^3}$ **b)** $\sqrt[3]{(8.024)^2}$ **c)** $\sqrt{1+(1.9)^3}$ **d)** $\cos(0.01)\cdot e^{-0.01}$ **e)** $0.02\cdot e^{0.02}$ **f)** $\frac{1.9}{4.2}$ **g)** $\frac{102}{98}$ **h)** $\sqrt[3]{(2.96)^2-1}$ **i)** $\ln(3-\sqrt{4.02})$ **j)** $\sqrt{\frac{1.04}{3.96}}$

E8 — derivative in one step

Calculate in one step the derivative (do not simplify the result) — see examples on PRE. If the derivative is calculated in several steps, we check only the formula after last equality sign.

Exam — theory

Test consists of about 15–20 tasks, identical or similar to tasks listed below. The pass mark is 24 out of 60 points.

1. arcsin — definition, graph, properties (i.e.: domain, range, zeros, monotonicity, where it is continuous, differentiable? is it bounded, concave up/down?).
Analogously for arccos, arctan, arccot , ln, sinh, cosh.

2. Definition of a metric/metric space.

3. Definition of cluster point/isolated point/interior point/open set/closed set/bounded set.

4. Definition of convergent sequence and its limit.

5. State and prove the Squeeze Theorem/ theorem on reduction of the convergence to the case of real sequence in Euclidean metric space/ theorem on the uniqueness of the limit/ theorem on bounded and monotonic sequence.

6. Derive the formula for: $\lim\limits_{n\to\infty}\sqrt[n]{n}$, $\lim\limits_{n\to\infty}\frac{a^n}{n!}$.

7. Give the definition for $\lim\limits_{x\to+\infty} f(x) = a$.

$\lim\limits_{x\to+\infty} f(x) = a \iff$ ....

where $f\colon D\to$ ....., $D\subset$ ....., $a\in$ ....., D is ...., and...
Make a figure.
Analogously for other limits.

8. Complete the definition: $\lim\limits_{x\to\dots} f(x) = \dots \iff \forall_m \exists_{\delta>0} [\forall_{x\in D}\ 0 < d_X(x,a) < \delta \Rightarrow f(x) < m]$

where $f\colon D\to$ ....., $D\subset$ ....., $a\in$ ....., a is ...., and .... are metric spaces. Make a figure.

9. Complete the definition: $\lim\limits_{x\to\dots} f(x) = \dots \iff \forall_{\{x_n\}} \left[(x_n\in D,\ x_n < b,\ \lim\limits_n x_n = b) \Rightarrow \lim\limits_n f(x_n) = a\right]$

where $f\colon D\to$ ....., $D\subset$ ....., $a\in$ ....., $b\in$ ......, ... is the cluster point of ....., and .... are metric spaces. Make a figure.

10. Derive the formula for: $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$, $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x$, $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$, $\lim_{x\to 0}\frac{a^x-1}{x}$

11. Find directly from the definition the derivative of $f(x)=\sin x$

Analogously for $f(x)=a^x$, $f(x)=e^x$, $f(x)=\ln x$, $f(x)=\frac{1}{x}$, $f(x)=x^2$, $f(x)=x^3$, $f(x)=\sqrt{x}$.

12. The Lagrange Theorem/ the Fermat Theorem/ the Rolle Theorem — state and prove.

13. 1st/2nd/3rd/4th corollary from the Lagrange Theorem — state and prove.

14. Relations between monotonicity/concavity and the sign of appropriate derivative.

15. The derivative of a function at a point — definition and geometrical interpretation.

16. Differential — definition and geometrical interpretation.

17. Derive the formula for the differential of sum/difference/product/quotient of two differentiable functions u and v.

18. Prove that if $y=f(x)$ is a differentiable function, then its increment may be written in the form $\Delta y = dy+\alpha\Delta x$, where $\lim_{\Delta x\to 0}\alpha=0$.

19. Extreme values — definitions, necessary and sufficient conditions, corollary from the Taylor Formula.

20. "If a function is continuous at a, then it is differentiable at a" — it is .... (true/false) statement because....... (justify your answer, i.e. prove or give a counterexample).
Analogously for statements:

a) If a sequence is convergent, then it is bounded.
b) If a sequence is divergent, then it is unbounded.
c) If a sequence is bounded, then it is convergent.
d) If a decreasing sequence is bounded, then it is convergent.
e) If a sequence is monotonic, then it is convergent.
f) If f is differentiable at x_0, then f is continuous at x_0.
g) If f is discontinuous at x_0, then f is not differentiable at x_0.
h) If f is not differentiable at x_0, then f is discontinuous at x_0.
i) Every function continuous at some point is differentiable at this point.
j) Every function differentiable at some point is continuous at this point.
k) There are functions continuous and not differentiable at a point.
l) There are functions discontinuous and differentiable at a point.
m) Continuity is necessary for differentiability.
n) Continuity is sufficient for differentiability.
o) Differentiability is necessary for continuity.
p) Differentiability is sufficient for continuity.
q) If f is defined on $\mathbb{R}$, then its graph has no vertical asymptotes.
r) There is a function f such that the curve $y=f(x)$ has simultaneously vertical, horizontal and slant asymptotes such that the slant asymptote is not parallel to the x–axis.

21. Complete

1) $\lim\limits_{t\to 0} \dfrac{7^t-1}{t}$ =..........

2) $\lim\limits_{t\to 0} \dfrac{2^t-t}{t}$ =..........

3) $\lim\limits_{t\to 0} \dfrac{t}{1-3^t}$ =..........

4) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ =..........

5) $\lim\limits_{x\to +\infty} \dfrac{\sin x}{x}$ =..........

6) $\lim\limits_{x\to -\infty} \dfrac{\sin x}{x}$ =..........

7) $\lim\limits_{x\to 3} \dfrac{\sin x}{x}$ =..........

8) $\lim\limits_{x\to 2} \dfrac{\sin(x-2)}{x^2-2x}$ =..........

9) $\lim\limits_{x\to 0} \dfrac{\sin(x-2)}{x^2-2x}$ =..........

10) $\lim\limits_{n\to +\infty} \dfrac{\sin n}{n}$ =..........

11) $\lim\limits_{n\to +\infty} n \sin \dfrac{1}{n}$ =..........

12) $\lim\limits_{n\to +\infty} \sqrt[n]{n}$ =..........

13) $\lim\limits_{n\to +\infty} \sqrt[n]{5}$ =..........

14) $\lim\limits_{n\to +\infty} (\sqrt[n]{n} + \sqrt[n]{2})$ =..........

15) $\lim\limits_{n\to +\infty} \left(1-\dfrac{1}{n}\right)^n$ =..........

16) $\lim\limits_{n\to +\infty} \left(1+\dfrac{2}{n}\right)^n$ =..........

17) $\lim\limits_{n\to +\infty} \left(1-\dfrac{1}{n}\right)^{3n}$ =..........

18) $\lim\limits_{x\to 0} e^x$ =..........

19) $\lim\limits_{x\to 0} e^{-x}$ =..........

20) $\lim\limits_{x\to +\infty} e^x$ =..........

21) $\lim\limits_{x\to -\infty} e^x$ =..........

22) $\lim\limits_{x\to 0^+} \ln x$ =..........

23) $\lim\limits_{x\to 1} \ln x$ =..........

24) $\lim\limits_{x\to +\infty} \ln x$ =..........

25) $\lim\limits_{x\to +\infty} \arctan x$ =..........

26) $\lim\limits_{x\to -\infty} \arctan x$ =..........

27) $\lim\limits_{x\to 0} \arctan x$ =..........

28) $\lim\limits_{x\to 1} \arctan x$ =..........

29) $\lim\limits_{x\to +\infty} \operatorname{arccot} x$ =..........

30) $\lim\limits_{x\to -\infty} \operatorname{arccot} x$ =..........

31) $\lim\limits_{x\to -1} \operatorname{arccot} x$ =..........

32) $\lim\limits_{x\to 0} \operatorname{arccot} x$ =..........