PART 1
INTRODUCTION

Sets of numbers

$\mathbb{N} = \{1,2,3,\cdots\}$ set of natural numbers

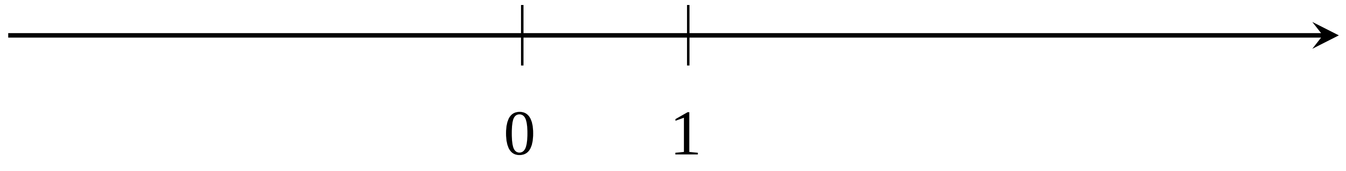
$\mathbb{Z} = \{0,\pm 1,\pm 2,\pm 3,\cdots\}$ set of integer numbers

$\mathbb{Q} = \left\{\frac{a}{b}:\ a \in \mathbb{Z} \land b \in \mathbb{N}\right\}$ set of rational numbers

$\mathbb{R} = \mathbb{Q} \cup I\mathbb{Q}$ set of real numbers

$\mathbb{C} = \{a + bi:\ a, b \in \mathbb{R}, i^2 = -1\}$ set of complex numbers

The real line (or axis) – a line with indicated direction, origin 0 and unit 1.



Any real number has its own, one and only one, position on the real line, and vice versa: any point on the real line corresponds to exactly one real number.

Intervals – subsets of $\mathbb{R}$ of the form:

$$\{x \in \mathbb{R}\colon\ a < x < b\} = (a, b)$$
$$\{x \in \mathbb{R}\colon\ a \leq x \leq b\} = \langle a, b\rangle$$
$$\{x \in \mathbb{R}\colon\ a \leq x < b\} = \langle a, b)$$
$$\{x \in \mathbb{R}\colon\ a < x \leq b\} = (a, b\rangle$$

$$\{x \in \mathbb{R}\colon\ x > a\} = (a, +\infty)$$
$$\{x \in \mathbb{R}\colon\ x \geq a\} = \langle a, +\infty\rangle$$
$$\{x \in \mathbb{R}\colon\ x < b\} = (-\infty, b)$$
$$\{x \in \mathbb{R}\colon\ x \leq b\} = (-\infty, b\rangle$$

Note:

$\mathbb{R} = (-\infty, +\infty)$, $\quad (4,0) = \emptyset$, $\quad \langle 3,3\rangle = \{3\}$

Df. 1. We say that set $S \subset \mathbb{R}$ is bounded above iff

$$\exists_{M\in\mathbb{R}}\forall_{x\in S}\ x \leq M.$$

Number M is called the upper bound of S.

Df. 2. We say that set $S \subset \mathbb{R}$ is bounded below iff

$$\exists_{M\in\mathbb{R}}\forall_{x\in S}\ x \geq M.$$

Number M is called the lower bound of S.

Df. 3. Set S is bounded iff it is bounded above and below.

Example: $S = \left\{\frac{1}{n}:\ n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\right\}$

Set S is bounded above (by 1, 20, $\sqrt{5}$, etc.).

Set S is bounded below (by –2, 0, -15, etc.).

Hence S is bounded.

Note that 1is the least upper bound (i.e. supremum of S) and 0 is the greatest lower bound (i.e. infimum of S).

We write:

$$\inf S = 0, \qquad \sup S = 1.$$

Note that LUB and GUB are not necessarily members of *S*.

Example: $S = \{x \in \mathbb{N}\colon\ x^2 < 5\}$

$\inf S = ?$ $\sup S = ?$

Example: $S = \{x \in \mathbb{Q}\colon\ x^2 < 5\}$

$\inf S = ?$ $\sup S = ?$

Completness Axiom:

Any nonempty bounded above subset of $\mathbb{R}$ has the supremum.

It follows from CA that any bounded below nonempty subset of $\mathbb{R}$ *has the infimum. Therefore CA is an expression of the fact that there are no gaps or holes on the real line.*

Cartesian product

Df. 4. The Cartesian product of sets A and B is the set of all ordered pairs such that the first element of the pair belongs to A and the second one belongs to B, i.e.:

$$A \times B = \{(a, b)\colon\ a \in A \ \ \wedge \ \ b \in B\}.$$

Example: if $A = \{0,1,2\}$ and $B = \{x, y\}$, then:

$$A \times B = \{(0, x), (0, y), (1, x), (1, y), (2, x), (2, y)\}$$

$$B \times A = \{(x, 0), (x, 1), (x, 2), (y, 0), (y, 1), (y, 2)\}$$

$$B \times B = \{(x, x), (x, y), (y, x), (y, y)\}$$

In general, the Cartesian product is not commutative.

$A \times A = A^2 \qquad A \times \emptyset = \emptyset \qquad \emptyset \times A = \emptyset$

Analogously,

$A \times B \times C = \{(a, b, c) \colon \ a \in A \ \wedge \ b \in B \ \wedge \ c \in C\}.$

$A \times A \times A = A^3$ and so on.

Example: give the geometrical interpretation of

a) $(1,2) \times \langle -1, -1 \rangle$

b) $(1,2) \times \{-1,1\}$

c) $\mathbb{N} \times \langle 0,1 \rangle$

d) $\{2\} \times \mathbb{R}$

e) $\mathbb{N} \times \mathbb{R}$

f) $\mathbb{R}^2$

g) $\mathbb{R}^3$

h) $\mathbb{R}^2 \times \{0\}$

i) $\{0\} \times \mathbb{R}^2$

About theorems

Usually theorems have the form ‘if *p*, then *q*’

$$p \Rightarrow q$$

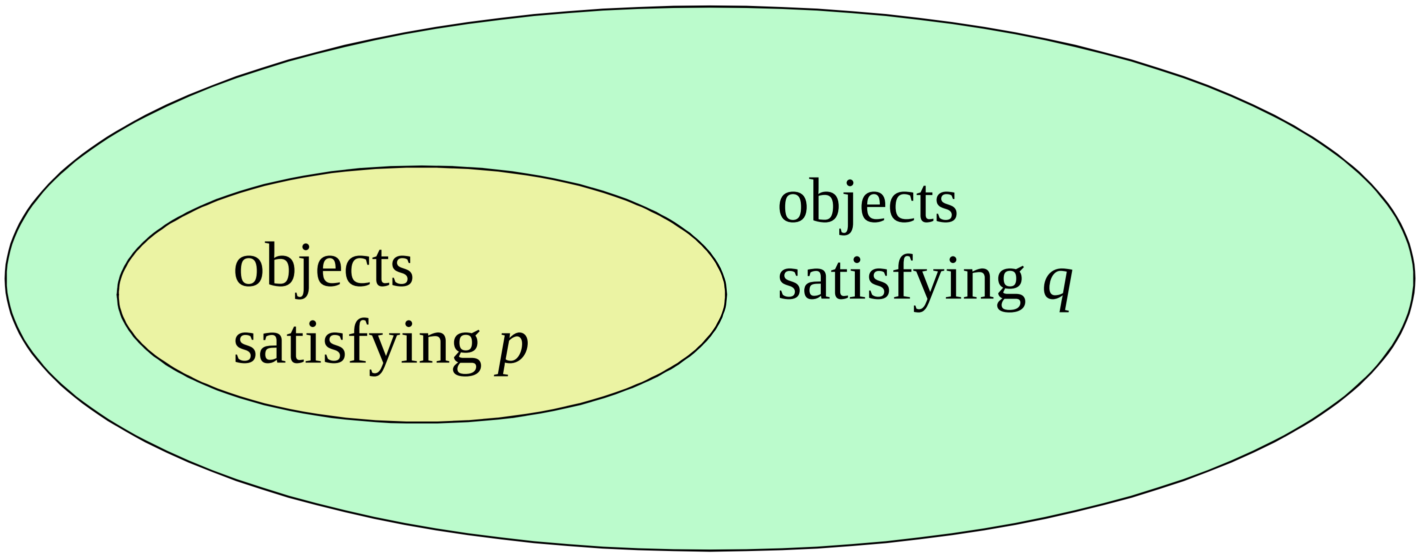
The converse of above theorem, i.e. ‘if *q*, then *p*’ may be a false statement.

Example:

$x > 0 \Rightarrow x^2 > 0$ TRUE

$x^2 > 0 \Rightarrow x > 0$ FALSE

$p \Rightarrow q$ $(\neg q) \Rightarrow (\neg p)$



p is a **sufficient** condition for *q*

q is a **necessary** condition for *p*

If both sentences $p \Rightarrow q$ and $q \Rightarrow p$ are true, then we can write

$$p \Leftrightarrow q$$

(*p* if and only if *q*).

Therefore *p* is necessary and sufficient for *q*, and vice versa.

Absolute value

Df. 5. The absolute value (modulus) of real number x is defined as follows

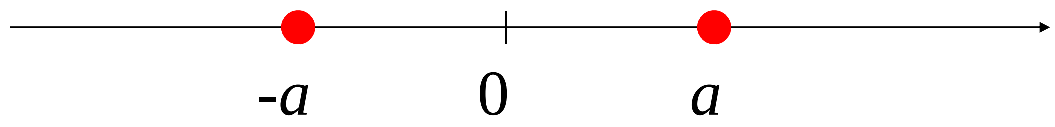
$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0. \end{cases}$$

Note that the modulus of x is the distance from x to 0.

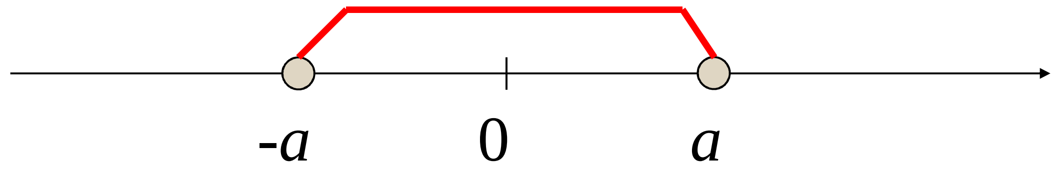
Example:

Let $a > 0$.

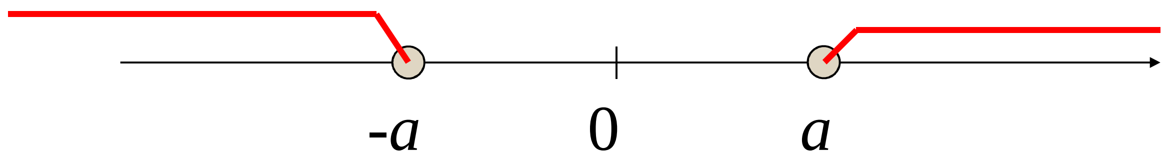
$$|x| = a \Leftrightarrow [x = a \vee x = -a]$$



$$|x| < a \Leftrightarrow -a < x < a$$



$$|x| > a \Leftrightarrow [x > a \vee x < -a]$$



Example:

$$|6-2|=4 \qquad |-3-2|=5$$

$$|4-12|=8 \qquad |-3-5|=8$$

$$|6-(-1)|=7 \qquad |-4-(-3)|=1$$

$$|1-(-5)|=6 \qquad |-4-(-7)|=3$$

$|a-b|=$ distance from a to b