# PART 10
# INDEFINITE INTEGRALS (2)

We can integrate any rational function (by long division and decomposition into partial fractions).

Some integrals, after an appropriate substitution called rationalizing substitution, become integrals of rational functions.

# Integration of trigonometric functions

**I** $\int R(\sin x, \cos x)dx$

$R$ – a rational function of two variables: $\sin x$ and $\cos x$

Universal trigonometric substitution: $t = \tan\frac{x}{2}$

$$\frac{x}{2} = \arctan t \Leftrightarrow x = 2\arctan t \Leftrightarrow dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2} = \int R_1(t)dt$$

Examples:

$$1)\ \int \frac{dx}{\sin x} = \left[ \begin{array}{c} t = \tan\frac{x}{2} \\ dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} \end{array} \right] = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} =$$

$$= \ln|t| + C = \ln\left|\tan\frac{x}{2}\right| + C$$

$$2)\ \int \frac{(1+\tan x)dx}{3\sin^2 x-\cos x-2}=\begin{bmatrix} t=\tan\frac{x}{2} & \cos x=\frac{1-t^2}{1+t^2} \\ dx=\frac{2dt}{1+t^2} & \tan x=\frac{2t}{1-t^2} \\ \sin x=\frac{2t}{1+t^2} & \end{bmatrix}=$$

$$=\int \frac{\left(1+\frac{2t}{1-t^2}\right)\frac{2dt}{1+t^2}}{3\left(\frac{2t}{1+t^2}\right)^2-\frac{1-t^2}{1+t^2}-2}=\int \frac{2(1+2t-t^2)(1+t^2)}{(1-t^2)(-3+8t^2-t^4)}dt=\dots$$

**II**

$$\int R(\sin x, \cos^2 x)\cos x dx =$$

$$= \begin{bmatrix} t = \sin x \\ dt = \cos x dx \\ \cos^2 x = 1 - t^2 \end{bmatrix} = \int R\left(t, 1-t^2\right)dt = \int R_1(t)dt$$

**III**

$$\int R(\sin^2 x, \cos x)\sin x dx =$$

$$= \begin{bmatrix} t = \cos x \\ dt = -\sin x dx \\ \sin^2 x = 1 - t^2 \end{bmatrix} = \int -R\left(1-t^2, t\right)dt = \int R_1(t)dt$$

Example:

$$\int \frac{\cos x dx}{2+2\sin x+\cos^2 x} = \begin{bmatrix} t=\sin x \\ dt=\cos x dx \\ \cos^2 x = 1-t^2 \end{bmatrix} =$$

$$= \int \frac{dt}{2+2t+1-t^2} = \int \frac{dt}{-t^2+2t+3} = \int \frac{dt}{-(t+1)(t-3)} =$$

$$= \int \left( \frac{\frac{1}{4}}{t+1} - \frac{\frac{1}{4}}{t-3} \right) dt = \frac{1}{4}\left[\ln|t+1| - \ln|t-3|\right] + C = \frac{1}{4}\ln\left|\frac{t+1}{t-3}\right| + C =$$

$$= \frac{1}{4}\ln\left|\frac{\sin x+1}{\sin x-3}\right| + C = \frac{1}{4}\ln\frac{\sin x+1}{3-\sin x} + C.$$

Example:

$$\int \frac{dx}{\sin x} = \int \frac{\sin x dx}{\sin^2 x} = \int \frac{\sin x dx}{1-\cos^2 x} = \begin{bmatrix} t = \cos x \\ dt = -\sin x dx \end{bmatrix} =$$

$$= \int \frac{-dt}{1-t^2} = \int \frac{-dt}{(1-t)(1+t)} = \int \left( \frac{\frac{1}{2}}{t-1} - \frac{\frac{1}{2}}{t+1} \right) dt =$$

$$= \frac{1}{2}\left[\ln|t-1| - \ln|t+1|\right] + C = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2}\ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

Note: in this integral the UTS was better (faster).

**IV** $$\int R(\sin^2 x, \cos^2 x, \sin x \cos x)dx$$

$$t = \tan x \Leftrightarrow x = \arctan t \Leftrightarrow dx = \frac{dt}{1+t^2}$$

$$\sin^2 x = \frac{t^2}{1+t^2}, \quad \cos^2 x = \frac{1}{1+t^2}, \quad \sin x \cos x = \frac{t}{1+t^2}$$

$$\int R(\sin^2 x\,,\ \cos^2 x\,,\ \sin x \cos x)dx =$$

$$= \int R\left(\frac{t^2}{1+t^2}, \frac{1}{1+t^2}, \frac{t}{1+t^2}\right)\frac{dt}{1+t^2} = \int R_1(t)dt$$

# Integration of irrational functions

**I**

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx =$$

$$= \left[\begin{array}{l} t = \sqrt[n]{\dfrac{ax+b}{cx+d}} \\ t^n = \dfrac{ax+b}{cx+d} \\ x = r(t) \\ dx = r'(t)dt \end{array}\right] = \int R\big(r(t), t\big) r'(t)dt = \int R_1(t)dt$$

Example:

$$\int \frac{1}{x-1}\left(1+\sqrt{\frac{x}{x-1}}\right)dx = \left[\begin{array}{c} t = \sqrt{\frac{x}{x-1}} \\ t^2 = \frac{x}{x-1} \\ x = \frac{t^2}{t^2-1} \\ dx = \frac{-2tdt}{(t^2-1)^2} \end{array}\right] =$$

$$= \int \frac{1}{\frac{t^2}{t^2-1}-1}(1+t)\frac{-2tdt}{(t^2-1)^2} = \int \frac{-2tdt}{t-1} = \cdots$$

**II**

$$\int R\left(x, \sqrt[n_1]{x}, \sqrt[n_2]{x}, \ldots, \sqrt[n_k]{x}\right) dx =$$

$$= \begin{bmatrix} t = \sqrt[n]{x} \\ t^n = x \\ dx = nt^{n-1}dt \end{bmatrix} = \int R_1(t)\,dt$$

$n$ – the lowest common multiple of $n_1, n_2, \ldots, n_k$

Example:

$$\int \frac{\sqrt{x}-\sqrt[3]{x}}{x(\sqrt[4]{x}+\sqrt[6]{x})}dx = \begin{bmatrix} t=\sqrt[12]{x} \Rightarrow x=t^{12} \Rightarrow dx = 12t^{11}dt \\ \sqrt{x}=t^{6}, \sqrt[3]{x}=t^{4}, \sqrt[4]{x}=t^{3}, \sqrt[6]{x}=t^{2} \end{bmatrix} =$$

$$= \int \frac{t^{6}-t^{4}}{t^{12}(t^{3}+t^{2})} 12t^{11}dt = 12\int \frac{t^{3}-t}{t+1}dt = 12\int \left[t^{2}-t\right]dt =$$

$$= 12\left[\frac{t^{3}}{3}-\frac{t^{2}}{2}\right]+C = 4\sqrt[4]{x}-6\sqrt[6]{x}+C.$$

**III** $$\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$$

Euler's substitutions:

$$\sqrt{ax^2+bx+c} = xt \pm \sqrt{c} \qquad \text{if} \quad c > 0$$

$$\sqrt{ax^2+bx+c} = t \pm x\sqrt{a} \qquad \text{if} \quad a > 0$$

$$\sqrt{ax^2+bx+c} = t(x - x_1) \qquad \text{if} \ \Delta > 0, \ \ ax_1^2 + bx_1 + c = 0$$

Przykład:

$$\int \frac{dx}{\sqrt{x^2+K}} = \left[ \begin{array}{c} \sqrt{x^2+K} = t - x \\ x^2 + K = t^2 - 2tx + x^2 \\ x = \frac{t^2-K}{2t} \\ dx = \frac{t^2+K}{2t^2}dt \\ \sqrt{x^2+K} = t - x = t - \frac{t^2-K}{2t} = \frac{t^2+K}{2t} \end{array} \right] =$$

$$= \int \frac{\frac{t^2+K}{2t^2}dt}{\frac{t^2+K}{2t}} = \int \frac{dt}{t} = \ln|t| + C = \ln\left|x + \sqrt{x^2+K}\right| + C$$

**IV** $$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx$$

Method of undetermined coefficients:

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$\frac{P_n(x)}{\sqrt{ax^2+bx+c}} = Q'_{n-1}(x)\sqrt{ax^2+bx+c} +$$

$$+ Q_{n-1}(x)\frac{2ax+b}{2\sqrt{ax^2+bx+c}} + \frac{\lambda}{\sqrt{ax^2+bx+c}}$$

$$P_n(x) = \underbrace{Q'_{n-1}(x)(ax^2+bx+c) + Q_{n-1}(x)\left(ax+\frac{b}{2}\right) + \lambda}$$

a polynomial of *n*th degree

- we compare coefficients in the same powers of x,
- solve the system in unknown coefficients of $Q$ and $\lambda$
- substitute into formula

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

- calculate the integral on RHS

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

$a > 0$

appropriate substitution

$a < 0$

$$\int \frac{dt}{\sqrt{t^2 + K}}$$

$$\int \frac{dt}{\sqrt{1 - t^2}}$$

Example:

$$\int \sqrt{x^2 + K}\,dx =$$

$$= \int \frac{x^2 + K}{\sqrt{x^2 + K}} dx = (Ax + B)\sqrt{x^2 + K} + \lambda \int \frac{dx}{\sqrt{x^2 + K}}$$

$$A = \frac{1}{2},\ B = 0,\ \lambda = \frac{K}{2}$$

$$\int \sqrt{x^2 + K}\,dx = \frac{x}{2}\sqrt{x^2 + K} + \frac{K}{2}\int \frac{dx}{\sqrt{x^2 + K}} =$$

$$= \frac{x}{2}\sqrt{x^2 + K} + \frac{K}{2}\ln\left|x + \sqrt{x^2 + K}\right| + C.$$