PART 11
DEFINITE INTEGRAL

Definition of definite integral

We consider a bounded function $f : \langle a, b \rangle \to \mathbf{R}$

n – fixed natural number

$$P_n = \{\langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle, \langle x_2, x_3 \rangle, \ldots, \langle x_{n-1}, x_n \rangle\}$$

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

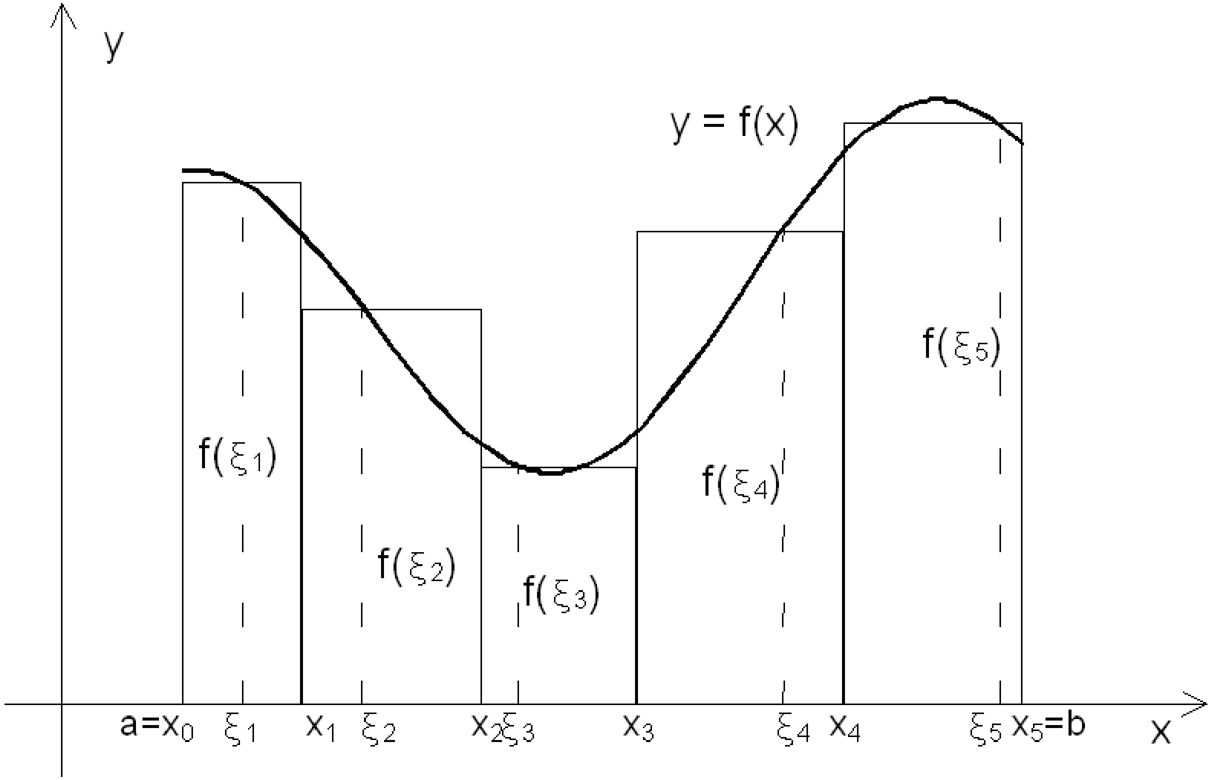
$$\Delta x_i = x_i - x_{i-1}, \quad i = 1, 2, \ldots, n$$

$$\lambda_n = \max_{1 \le i \le n} \Delta x_i$$ norm (mesh) of partition P_n

$$\xi_i \in \langle x_{i-1}, x_i \rangle, \quad i = 1, 2, \ldots, n$$ intermediate points (arbitrarily chosen)

the Riemann sum:

$$R_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots f(\xi_n)\Delta x_n$$



the Riemann sum depends on ...

The sequence $\{P_n\}$ is called a normal sequence of partitions iff

$$\lim_{n\to\infty} \lambda_n = 0$$

For each new partition we form the Riemann sum and then calculate the limit of obtained sequence $\{R_n\}$.

How one can interpret the limit of $\{R_n\}$?

If for all normal sequences of partitions of interval $\langle a,b\rangle$ the limit of sequence of Riemann's sums always exists and always has the same finite value, independently of how points ξ_i were chosen, then this limit is called the definite integral of f from a to b and it is denoted by

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ (\lambda_n\to 0)}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i .$$

a – lower limit of integration, b – upper limit of integration

Convention:

$$\int_a^a f(x)dx = 0$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Examples

1) $\int_a^b dx = ?$

$$\forall_x \quad f(x) = 1 \Rightarrow \forall_i \quad f(\xi_i) = 1$$

$$\lim_n R_n = \lim_n \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \lim_n \sum_{i=1}^{n} 1 \cdot \Delta x_i = \lim_n \sum_{i=1}^{n} \Delta x_i =$$

$$= \lim_n (b-a) = b-a$$

Therefore $\int_a^b dx = b - a.$

$$2)\ f(x)=\begin{cases}1 & \text{for} \quad x\in \mathbf{Q} \\ 0 & \text{for} \quad x\notin \mathbf{Q}\end{cases}, \qquad \int_a^b f(x)dx = ?$$

1^{o} $\xi_i \in \mathbf{Q},\ f(\xi_i)=1$

$$\lim_{n} R_n = \lim_{n} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \lim_{n} \sum_{i=1}^{n} 1\cdot \Delta x_i = b-a$$

2^{o} $\xi_i \notin \mathbf{Q},\ f(\xi_i)=0$

$$\lim_{n} R_n = \lim_{n} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \lim_{n} \sum_{i=1}^{n} 0\cdot \Delta x_i = \lim_{n} 0 = 0$$

The limit of sequence of Riemann's sums depends on particular choices of intermediate points so the function is not integrable.

Properties of definite integrals

Theorem 1. If function f is continuous on $\langle a,b \rangle$, then it is integrable over $\langle a,b \rangle$.

Proof:

P_n (n – fixed)

$$\forall_{1 \le i \le n} \; f \text{ is cont. on } \langle x_{i-1}, x_i \rangle$$

⇓ the Weierstrass Theorem

in the ith subinterval the function attains its global minimum m_i and global maximum M_i

$$m_i \le f(\xi_i) \le M_i, \quad i = 1,2,\ldots,n$$

$$m_i \Delta x_i \le f(\xi_i)\Delta x_i \le M_i \Delta x_i, \quad i = 1,2,\ldots.,n$$

$$\underbrace{\sum_{i=1}^{n} m_i \Delta x_i}_{s_n} \le \sum_{i=1}^{n} f(\xi_i)\Delta x_i \le \underbrace{\sum_{i=1}^{n} M_i \Delta x_i}_{S_n}$$

Let us consider a normal sequence of partitions of $\langle a, b \rangle$.

We obtain two sequences: $\{s_n\}$ i $\{S_n\}$, such that

$$s_1 \le s_2 \le s_3 \le \ldots. \le S_3 \le S_2 \le S_1$$

Sequences $\{s_n\}$ i $\{S_n\}$ are convergent (why?)

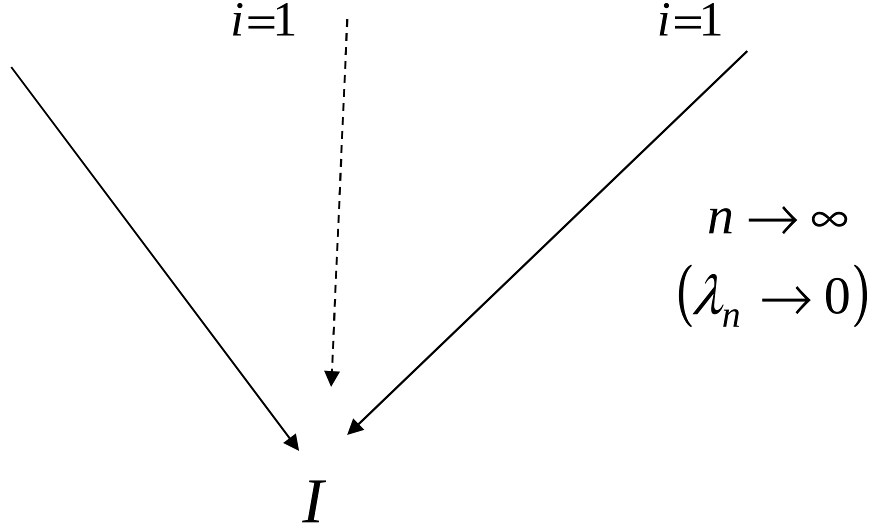
and they have the same limit:

$$0 \le \lim_{n\to\infty}\left(S_n - s_n\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(M_i - m_i\right)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n}\omega_i \Delta x_i \le$$

$$\le \lim_{n\to\infty}\sum_{i=1}^{n}\varepsilon_n \Delta x_i = \lim_{n\to\infty}\varepsilon_n \sum_{i=1}^{n}\Delta x_i = \lim_{n\to\infty}\varepsilon_n (b-a) = 0$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} s_n = I$$

$$\sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} f(\xi_i)\Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i$$



$$\lim_{\substack{n\to\infty \\ (\lambda_n \to 0)}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = I = \int_a^b f(x)\,dx.$$

Analogously, one can show that following functions are integrable:

- bounded and monotonic
- bounded with finite number of discontinuities.

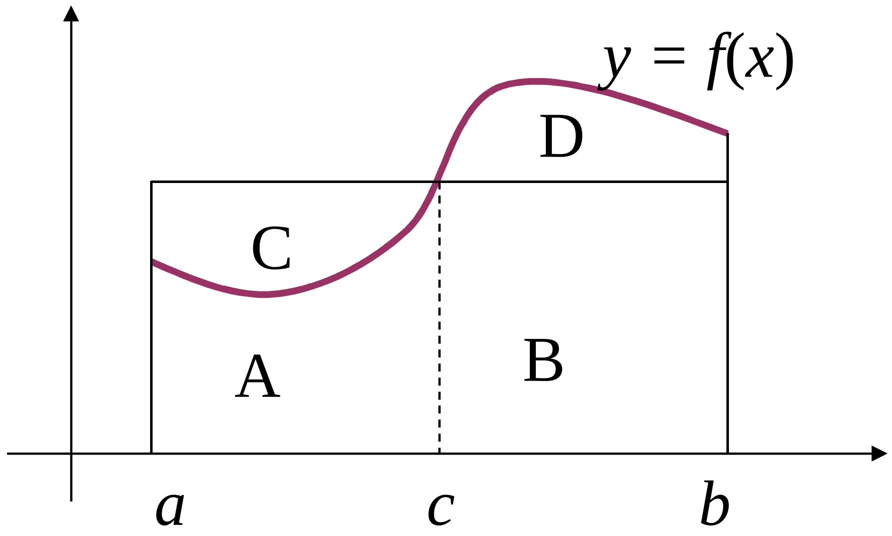
Theorem 2 (Mean-Value Theorem for definite integral).
If function f is continuous on $\langle a, b\rangle$, then there exists a number c in this interval such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Equivalently,

$$\int_a^b f(x)dx = (b-a)f(c).$$

Geometrical interpretation of Th. 2:



$$(b-a)f(c)=\int_a^b f(x)dx.$$

$$\int_a^b f(x)dx = A+B+D$$

$$(b-a)f(c)=A+B+C$$

$$C=D$$

Proof of Th. 2:

$$m \le f(\xi_i) \le M, \ \ i = 1,2,\dots,n$$

$$m\Delta x_i \le f(\xi_i)\Delta x_i \le M\Delta x_i, \quad i = 1,2,\dots.,n$$

$$\sum_{i=1}^{n} m\Delta x_i \le \sum_{i=1}^{n} f(\xi_i)\Delta x_i \le \sum_{i=1}^{n} M\Delta x_i$$

$$m(b-a) \le \sum_{i=1}^{n} f(\xi_i)\Delta x_i \le M(b-a)$$

$$\lim_{\substack{n\to\infty \\ (\lambda_n \to 0)}} m(b-a) \le \lim_{\substack{n\to\infty \\ (\lambda_n \to 0)}} \sum_{i=1}^{n} f(\xi_i)\Delta x_i \le \lim_{\substack{n\to\infty \\ (\lambda_n \to 0)}} M(b-a)$$

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$$

$\Downarrow$ f has the Darboux property

$$\exists_{c\in\langle a,b\rangle} \quad f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

Theorem 3. If functions f i g are integrable over $\langle a,b\rangle$, then

1) function $|f|$ is integrable over $\langle a,b\rangle$,

2) $$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

3) $$\int_a^b kf(x)dx = k\int_a^b f(x)dx \quad (k\in \mathbf{R})$$

4) $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \ \ a\le c\le b$$

5) $$\left[\forall_{x\in\langle a,b\rangle} f(x)\le g(x)\right] \Rightarrow \int_a^b f(x)dx \le \int_a^b g(x)dx$$

Theorem 4 (First Fundamental Theorem of Calculus).

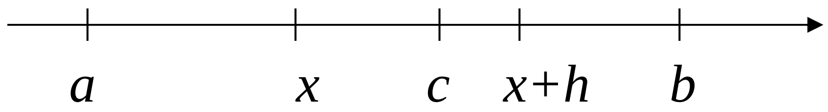
If function f is continuous on $\langle a,b\rangle$, then function

$$\varphi(x)=\int_a^x f(t)dt,\quad x\in\langle a,b\rangle$$

is continuous on $\langle a,b\rangle$, differentiable on (a,b), and

$$\forall_{x\in(a,b)}\quad \varphi'(x)=f(x).$$

Proof:



$1^{o} \quad h > 0$

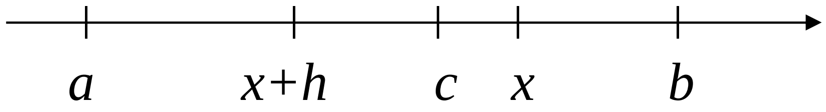
$$\varphi(x+h)-\varphi(x) = \int_a^{x+h} f(t)dt - \int_a^{x} f(t)dt =$$

$$\overset{\text{th.3.4}}{=} \int_a^{x} f(t)dt + \int_x^{x+h} f(t)dt - \int_a^{x} f(t)dt = \int_x^{x+h} f(t)dt \overset{\text{th.2}}{=} hf(c)$$

$$\lim_{h\to 0^+} [\varphi(x+h)-\varphi(x)] = \lim_{h\to 0^+} hf(c) = 0$$

bounded
(why?)

$2^{\text{o}} \quad h < 0$



$$\varphi(x+h) - \varphi(x) = \int_a^{x+h} f(t)dt - \int_a^{x} f(t)dt =$$

$$\overset{\text{th.3.4}}{=} \int_a^{x} f(t)dt - \int_a^{x} f(t)dt - \int_{x+h}^{x} f(t)dt = \int_x^{x+h} f(t)dt \overset{\text{th.2}}{=} hf(c)$$

$$\lim_{h\to 0^-} [\varphi(x+h) - \varphi(x)] = \lim_{h\to 0^-} hf(c) = 0$$

bounded

Since $\lim_{h\to 0^+}[\varphi(x+h)-\varphi(x)]=0$ and $\lim_{h\to 0^-}[\varphi(x+h)-\varphi(x)]=0$

we conclude $\lim_{h\to 0}[\varphi(x+h)-\varphi(x)]=0$

i.e. $\lim_{h\to 0}\varphi(x+h)=\varphi(x)$. Therefore function φ is continuous.

Function φ is differentiable:

$$\varphi'(x)=\lim_{h\to 0}\frac{\varphi(x+h)-\varphi(x)}{h}=\lim_{h\to 0}\frac{hf(c)}{h}=\lim_{\substack{(h\to 0)\\ c\to x}} f(c)=f(x).$$

■

It follows from First FFC:

$$\varphi(x) = \int_1^x \frac{\sin t}{t}\, dt \Rightarrow \varphi'(x) = \frac{\sin x}{x}$$

$$\varphi(x) = \int_x^0 e^{t^2}\, dt \Rightarrow \varphi'(x) = -e^{x^2}$$

These formulas hold for *x* such that ….

Theorem 4 (Second Fundamental Theorem of Calculus; the Newton-Leibniz Formula).

If function f is continuous on $\langle a,b \rangle$ and F is its antiderivative, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

Proof:

$$\varphi(x) = \int_a^x f(t)dt; \quad \varphi'(x) = f(x)$$

$$\varphi(a) = \int_a^a f(t)dt = 0$$

$$\varphi(b) = \int_a^b f(t)dt$$

$$F(x) = \varphi(x) + C$$

$$F(a) = \varphi(a) + C \qquad F(b) = \varphi(b) + C$$

$$F(a) = C \qquad F(b) = \int_a^b f(t)dt + C$$

$$F(b) = \int_a^b f(t)dt + F(a) \Rightarrow \int_a^b f(t)dt = F(b) - F(a).$$

∎

Examples:

1) $$\int_{-2}^{1} x^2 dx = \left.\frac{x^3}{3}\right|_{-2}^{1} = \frac{1^3}{3} - \frac{(-2)^3}{3} = 3$$

2) $$\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} \cos x dx = \sin x\Big|_{\frac{\pi}{3}}^{\frac{5\pi}{6}} = \sin\frac{5\pi}{6} - \sin\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2} = \frac{1-\sqrt{3}}{2}$$

3) $$\int_{0}^{\frac{\pi}{2}} \cos^2 x dx = I \qquad \int_{0}^{\frac{\pi}{2}} \sin^2 x dx = I$$

$$\int_{0}^{\frac{\pi}{2}} (\underbrace{\sin^2 x + \cos^2 x}) dx = 2I \qquad \frac{\pi}{2} = 2I \Rightarrow I = \frac{\pi}{4}$$

Theorem 5 (integration by parts). If $u, v \in C^1\langle a, b\rangle$, then

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)dx.$$

Example:

$$\int_0^1 \arctan x dx = \begin{bmatrix} u = \arctan x & v' = 1 \\ u' = \dfrac{1}{1+x^2} & v = x \end{bmatrix} = x \arctan x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}dx =$$

$$= 1 \cdot \arctan 1 - 0 \cdot \arctan 0 - \frac{1}{2}\ln\left(1+x^2\right)\Big|_0^1 =$$

$$= \frac{\pi}{4} - 0 - \frac{1}{2}\ln 2 + \frac{1}{2}\ln 1 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

Theorem 5 (Substitution Rule). If

1) function f is continuous on $\langle a, b \rangle$
2) functions φ i φ' are continuous on $\langle \alpha, \beta \rangle$
3) $\varphi(\alpha) = a, \ \varphi(\beta) = b$
4) the range of function $x = \varphi(t), \ t \in \langle \alpha, \beta \rangle$, is contained in $\langle a, b \rangle$

then

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt.$$

NOTE:

the limits of integration in Substitution Rule are changed.

Examples:

1) $$\int_0^1 (3x-1)^5 dx = \left[\begin{array}{c} t = 3x-1 \\ dt = 3dx \Rightarrow dx = \frac{1}{3}dt \\ \begin{array}{c|cc} x & 0 & 1 \\ \hline t & -1 & 2 \end{array} \end{array}\right] =$$

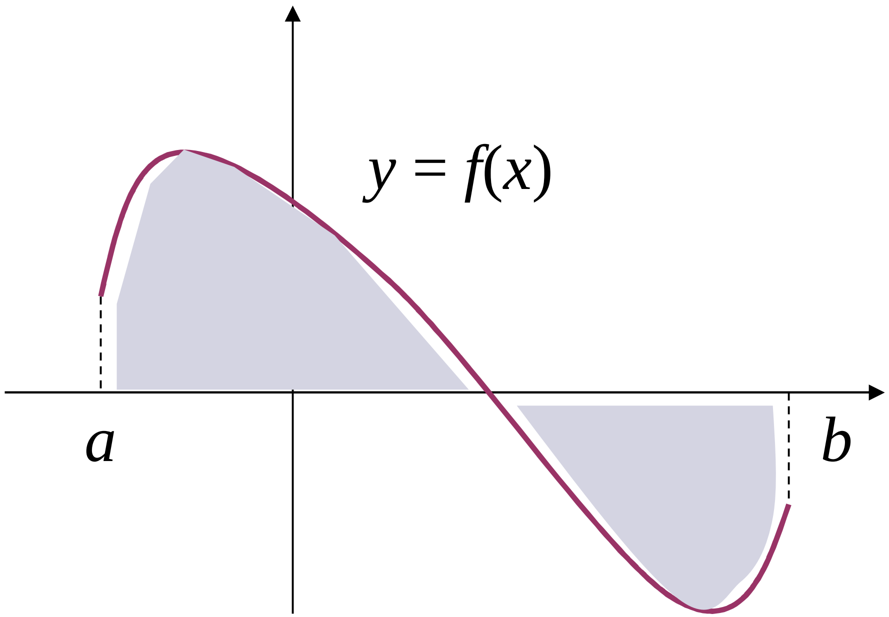
$$= \int_{-1}^{2} \frac{1}{3} t^5 dt = \frac{1}{3} \cdot \frac{t^6}{6}\Big|_{-1}^{2} = \frac{1}{18}\left[2^6 - (-1)^6\right] = \frac{63}{18}$$

2) $$\int_0^{\frac{\pi}{3}} \cos^2 x \sin x dx = \left[\begin{array}{c} t = \cos x \\ dt = -\sin x dx \\ \begin{array}{c|cc} x & 0 & \pi/3 \\ \hline t & 1 & 1/2 \end{array} \end{array}\right] =$$

$$= \int_1^{\frac{1}{2}} t^2 (-1) dt = -\frac{t^3}{3}\Big|_1^{\frac{1}{2}} = -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}$$

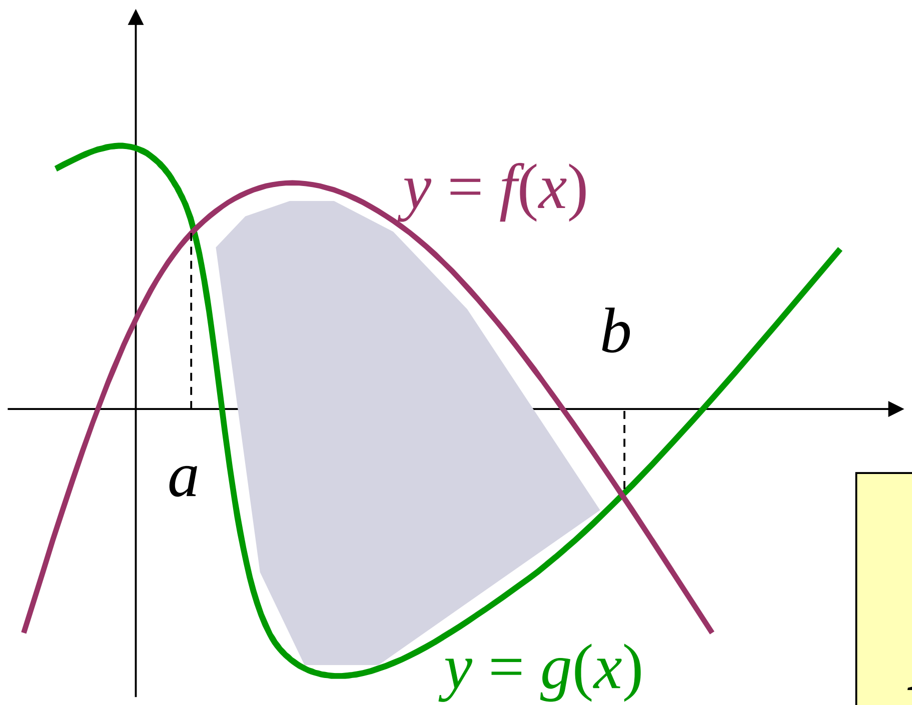
Applications of definite integral

I Area of a curvilinear trapezoid



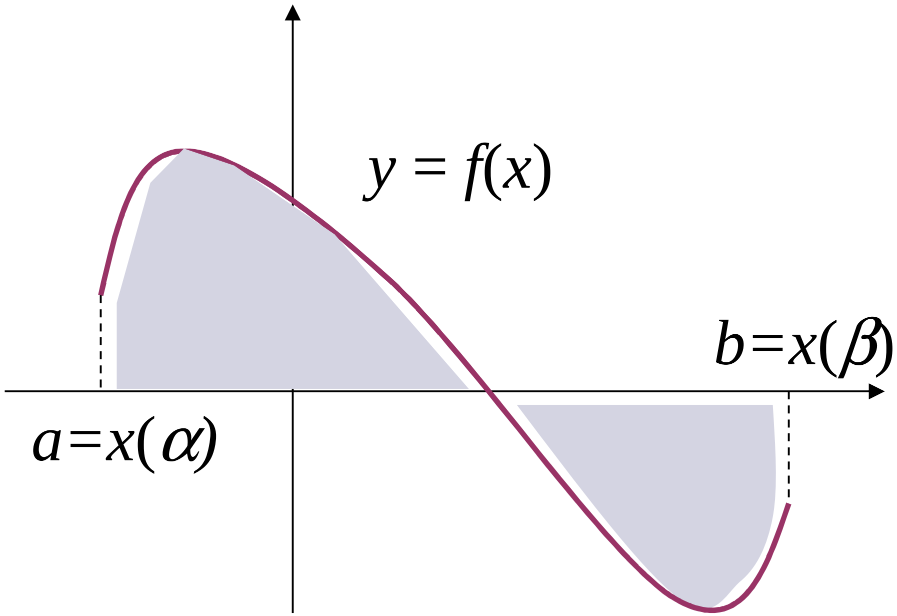
$$A = \int_a^b |f(x)| dx$$

II Area between two curves



$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

II Area under a parametric curve (1)



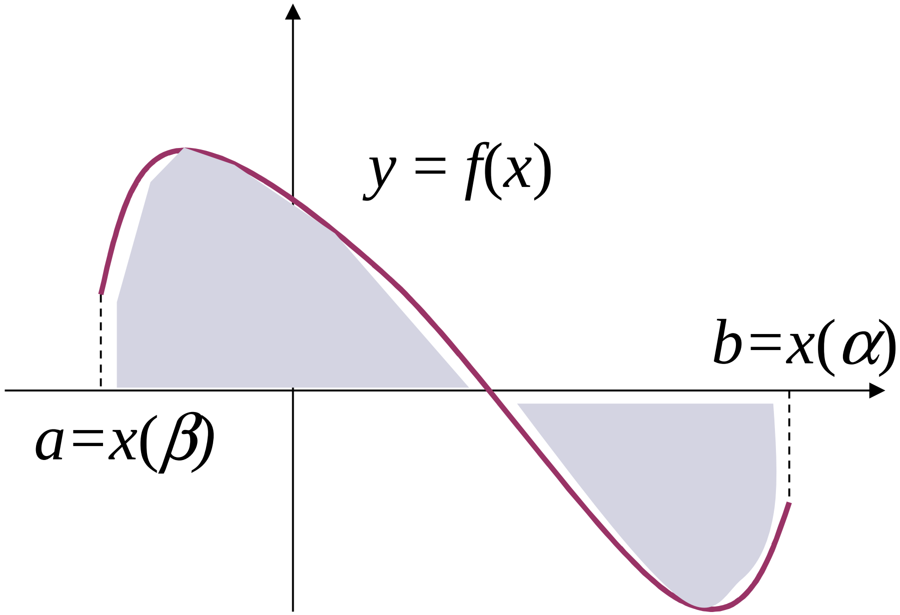
$$\begin{cases} x = x(t) \\ y = y(t),\ \alpha \le t \le \beta \end{cases}$$

$$x \in C^1\langle \alpha, \beta \rangle$$

$$y \in C\langle \alpha, \beta \rangle$$

$$A = \int_{\alpha}^{\beta} \left| y(t) \right| x'(t)\,dt$$

II Area under a parametric curve (2)



$$\begin{cases} x = x(t) \\ y = y(t),\ \alpha \le t \le \beta \end{cases}$$

$$x \in C^1\langle \alpha, \beta \rangle$$

$$y \in C\langle \alpha, \beta \rangle$$

$$A = \int_{\beta}^{\alpha} |y(t)| x'(t) dt$$

IV Arc length

$$y = f(x),\ x \in \langle a, b \rangle$$

$$f \in C^1 \langle a, b \rangle$$

$$l = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

IV Arc length of parametric curve

$$\begin{cases} x = x(t) \\ y = y(t), \ \alpha \le t \le \beta \end{cases} \qquad x, y \in C^1\langle \alpha, \beta \rangle$$

$$l = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$