PART 2
FUNCTIONS

Basic definitions

Df. 1. A function f from a set X to a set Y is a rule (or method) of assigning one and only one element in Y to each element in X. We write

$$f: X \rightarrow Y$$

The element that function f assigns to the element x is denoted $f(x)$. Then:

x – input/independent value/argument

$y = f(x)$ – output/dependent variable/value of f at x.

Note: f – function, $f(x)$ – value of function (not the same!)

X – domain (set of all inputs)

Y – codomain

R – range (set of all outputs)

It is possible that $Y \neq R$.

Df. 2. If a function $f: X \to Y$ takes on each value in set Y (i.e. $Y = R$), then f is called surjective (or a surjection, or onto function).

$$\forall_{y \in Y} \exists_{x \in X} \; y = f(x)$$

Examples:

$f: \mathbb{R} \to \mathbb{R}, \quad y = \sin x$ — not surjective

$f: \mathbb{R} \to \langle -1,1 \rangle, \quad y = \sin x$ — surjective

$f: (0, \pi) \to \langle -1,1 \rangle, \quad y = \sin x$ — not surjective

$f: (0, \pi) \to (0, 1\rangle \; y = \sin x$ — surjective

Df. 3. If a function $f: X \to Y$ sends distinct elements of X to distinct elements of Y, then f is called injective (or an injection, or one-to-one function).

$$\forall_{a,b\in X}\ [a \neq b \Longrightarrow f(a) \neq f(b)]$$

or, equivalently,

$$\forall_{a,b\in X}\ [f(a) = f(b) \Longrightarrow a = b].$$

Example:

$$f(x) = 2^{-x} - 2^{x}, x \in \mathbb{R}$$

$$f(a) = f(b)$$

$$2^{-a} - 2^{a} = 2^{-b} - 2^{b} \Big| \cdot 2^{a+b}$$

$$2^{b} - 2^{2a+b} = 2^{a} - 2^{a+2b}$$

$$2^{b} - 2^{a} - 2^{2a+b} + 2^{a+2b} = 0$$

$$(2^{b} - 2^{a}) + 2^{a+b}(2^{b} - 2^{a}) = 0$$

$$(2^{b} - 2^{a})(1 + 2^{a+b}) = 0$$

$$2^{b} - 2^{a} = 0$$

$$2^{b} = 2^{a}$$

$$a = b$$

f is injective

Example: $f(x)=\dfrac{x^2}{x+1},\ \ x\neq -1$

$$f(a)=f(b)$$

$$\frac{a^2}{a+1}=\frac{b^2}{b+1}$$

$$a^2(b+1)=b^2(a+1)$$

$$a^2b+a^2=b^2a+b^2$$

$$a^2b+a^2-b^2a-b^2=0$$

$$ab(a-b)+(a-b)(a+b)=0$$

$$(a-b)(ab+a+b)=0$$

$$a=b \ \vee \ ab+a+b=0$$

$$a(b+1)=-b$$

$$a=\frac{-b}{b+1}$$

$$f(1)=\frac{1}{1+1}=\frac{1}{2}$$

$$f(-0.5)=\frac{0.25}{-0.5+1}=\frac{0.25}{0.5}=\frac{1}{2}$$

f is not one-to-one

Df. 4. If a function f is both injective and surjective, then f is called bijective (or bijection, or one-to-one onto function).

Df. 5. If $f: X \to Y$ is bijective, then the inverse function f^{-1} is defined as

$$f^{-1}: Y \to X, \quad f^{-1}(y) = x \Leftrightarrow f(x) = y,$$

where $x \in X,\ y \in Y$.

Example:

$$f: y = 1 - \sqrt{x-2}, \qquad x \geq 2$$

$$\sqrt{x-2} = 1 - y$$
$$x - 2 = (1-y)^2$$
$$x = 2 + (1-y)^2$$

$$f^{-1}: y = x^2 - 2x + 3, \qquad x \leq 1$$

$$f^{-1}: x = y^2 - 2y + 3, \qquad y \leq 1$$

Corollaries (for real-valued functions of one real variable):

- If a point (a, b) belongs to the graph of function f, then point (b, a) belongs to the graph of function f^{-1}.
- Graphs of two mutually inverse functions are symmetric with respect to line $y = x$ (the bisector of the first quadrant).

Df. 6. If $f\colon X \to Y$ and $g\colon Y \to Z$, then the composition of g and f is the function $h\colon X \to Z$ such that $h(x) = g(f(x))$ for each x from X.

We write $h = g \circ f$ (f – inner function, g – outer function).

Example:

$$f(a) = a^2 + 1, g(b) = \sin b$$

$$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = \sin(x^2 + 1)$$

$$(f \circ g)(x) = f(g(x)) = f(\sin x) = \sin^2 x + 1$$

Here $g \circ f \neq f \circ g$.

Example:

$$h(x) = \sqrt{2x + 1}$$

inner function $f(x) = 2x + 1$,

outer function $g(x) = \sqrt{x}$.

Real-valued functions of one real variable

We consider functions of the type

$$f : D \to \mathbb{R},\ \ D \subset \mathbb{R}.$$

Df. 7. Function f is called periodic iff

$$\exists_{T \neq 0} \forall_{x \in D} \left[x + T \in D \wedge f(x+T) = f(x) \right].$$

Number T is then called a period of f. The least positive period of f is called the primitive period.

Give examples of two different periodic functions which do not have primitive periods.

Df. 8. A function f is increasing on a set $A \subset D$ iff

$$\forall_{x_1,x_2 \in A} [x_1 < x_2 \Rightarrow f(x_1) < f(x_2)].$$

Df. 9. A function f is decreasing on a set $A \subset D$ iff

$$\forall_{x_1,x_2 \in A} [x_1 < x_2 \Rightarrow f(x_1) > f(x_2)].$$

Df. 10. A function f is nonincreasing on a set $A \subset D$ iff

$$\forall_{x_1,x_2 \in A} [x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)].$$

Df. 11. A function f is nondecreasing on a set $A \subset D$ iff

$$\forall_{x_1,x_2 \in A} [x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)].$$

Examples:

1) Function $y = \frac{1}{x}$ is decreasing on $(-\infty, 0)$ and on $(0, +\infty)$. But it is not decreasing on its domain! We say that it is piecewise decreasing.

2) Similarly, function $y = \frac{1}{x^2}$ is piecewise monotonic.

3) Function $f(x) = x^4 - 2x^2 + 3$ is increasing on $(1, +\infty)$. Indeed, if $1 < x_1 < x_2$, then

$$f(x_1) - f(x_2) = x_1^4 - 2x_1^2 + 3 - x_2^4 + 2x_2^2 - 3 =$$

$$= (x_1^4 - x_2^4) - 2(x_1^2 - x_2^2) =$$

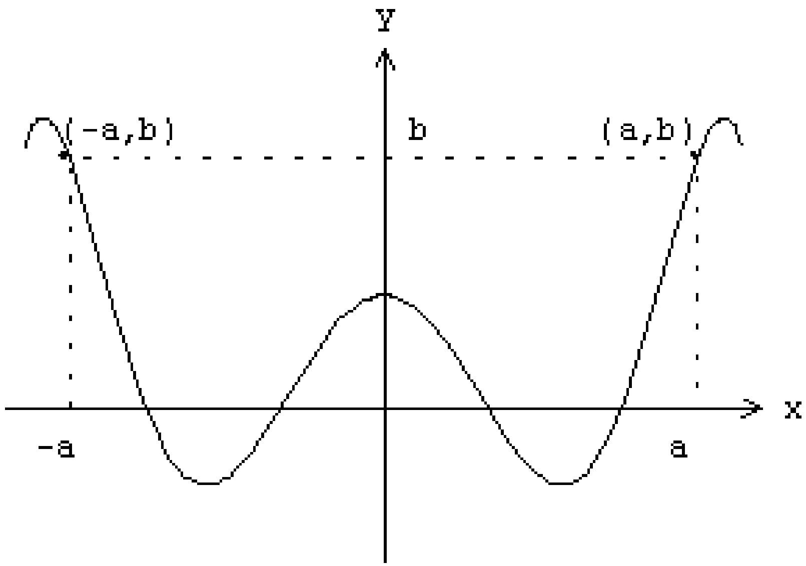
$$= (x_1^2 - x_2^2)(x_1^2 + x_2^2) - 2(x_1^2 - x_2^2) =$$

$$= (x_1^2 - x_2^2)(x_1^2 + x_2^2 - 2) < 0$$

so $f(x_1) < f(x_2)$.

Df. 12. Function f is called even iff

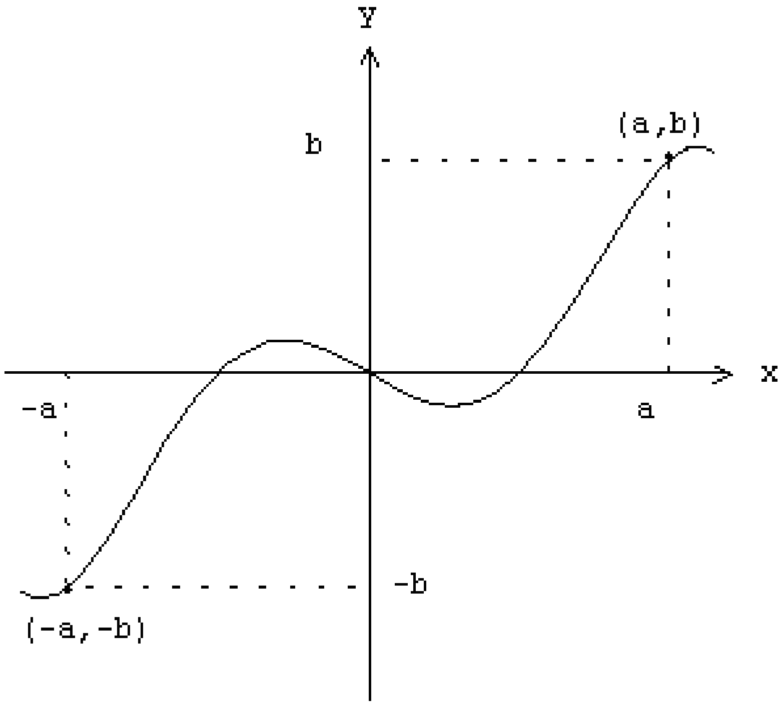
$$\forall_{x \in D} \left[-x \in D \wedge f(-x) = f(x) \right].$$



Its graph is symmetric with respect to *y*-axis.

Df. 13. Function f is called odd iff

$$\forall_{x\in D}\left[-x\in D \wedge f(-x)=-f(x)\right].$$



Its graph is symmetric with respect to the origin.

Df. 14. A function is bounded above (below) iff its range is bounded above (below).

If a function is bounded above and below then it is called bounded.

Elementary functions

Df. 15. The following four functions, defined on $\mathbb{R}$, are called basic elementary functions:

- unit function $U(x) = 1$,
- identity function $\mathrm{id}(x) = x$,
- exponential function $\exp(x) = e^x$,
- sine function $\sin(x) = \sin x$.

Df. 16. The following functions are called elementary functions:

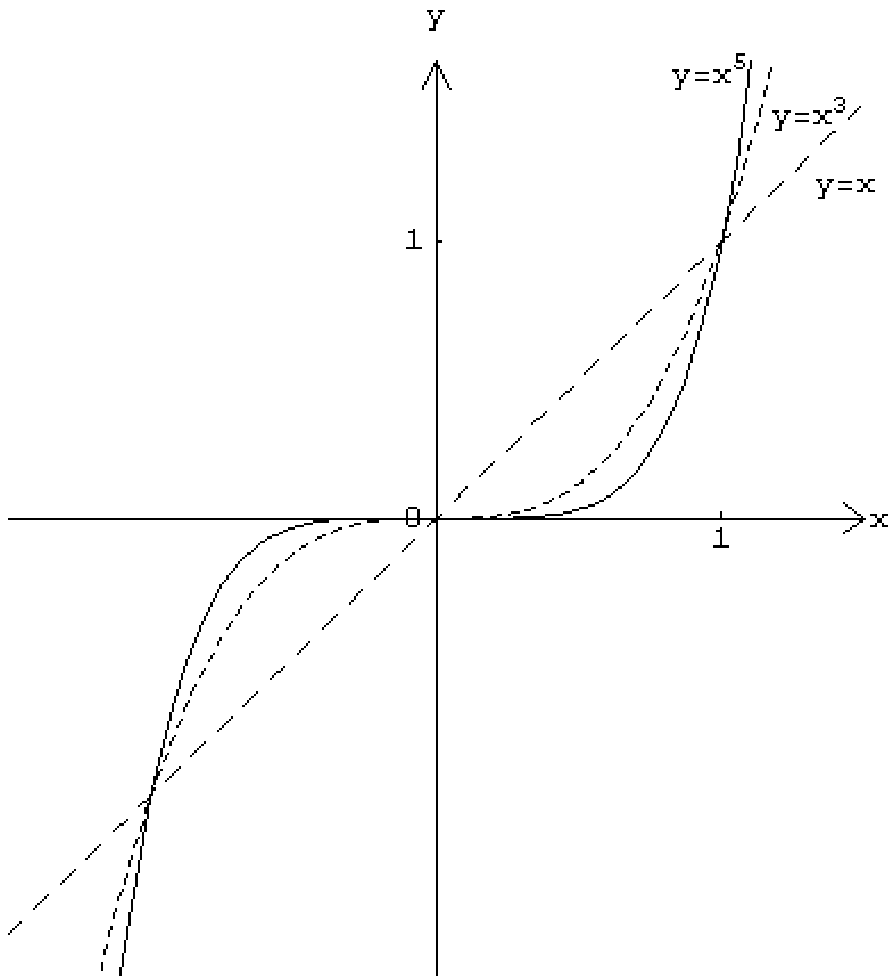
- each basic elementary function;
- constant multiple of elementary function;
- sum, difference, product, quotient of two elementary functions;
- the composition of two elementary functions;
- the inverse function to an elementary function;
- an elementary function with restricted domain

(of course if listed operations are feasible).

Examples of elementary functions

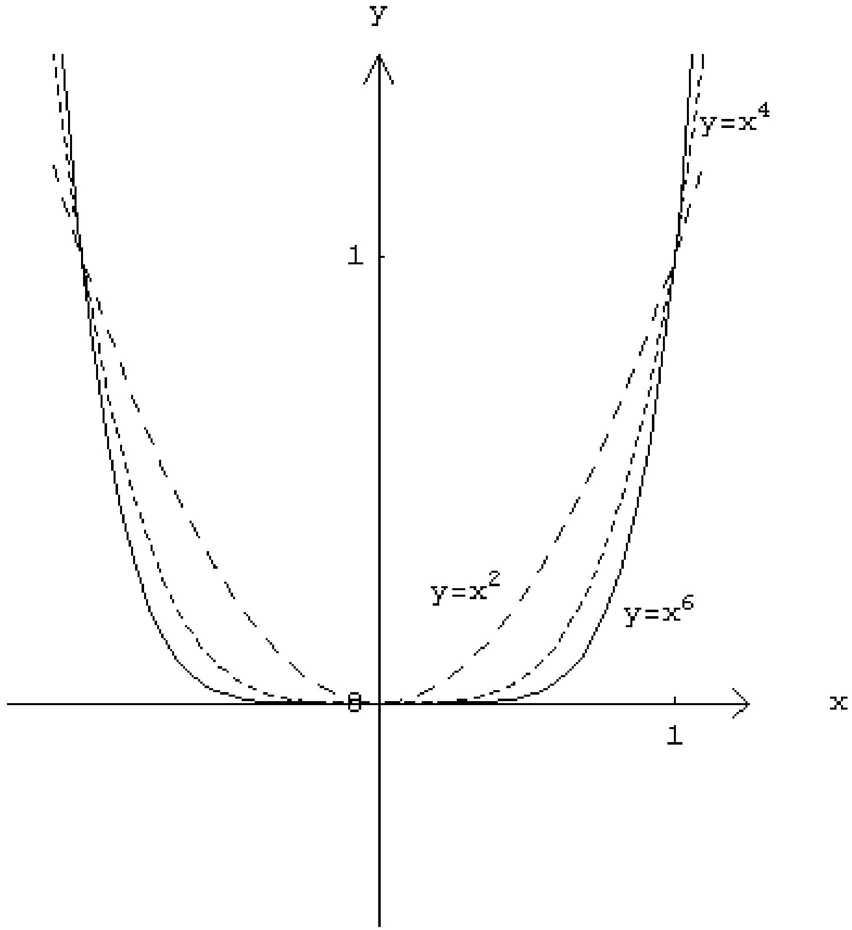
Power function

$y = x^n, n \in \mathbb{N}, n$ − odd



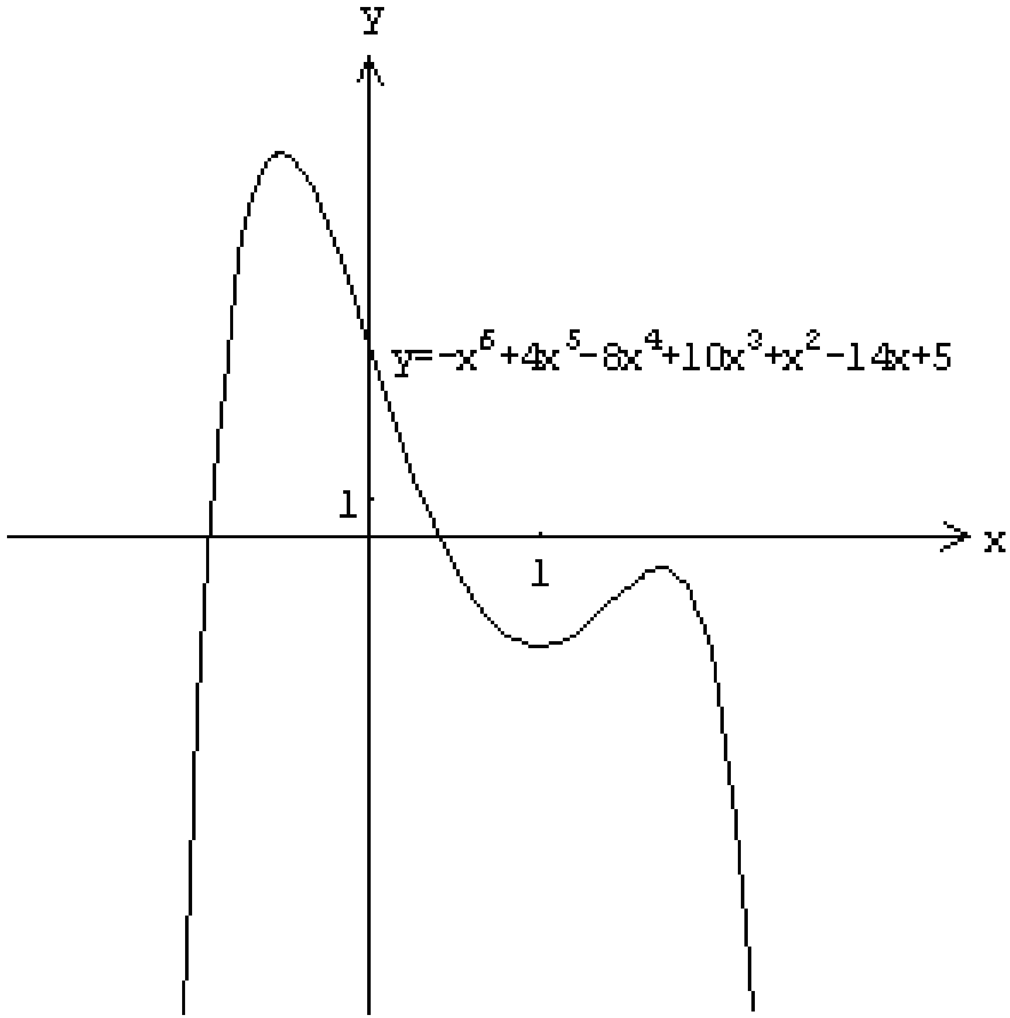
Examples of elementary functions

Power function
$y = x^n, n \in \mathbb{N}, n$ − even



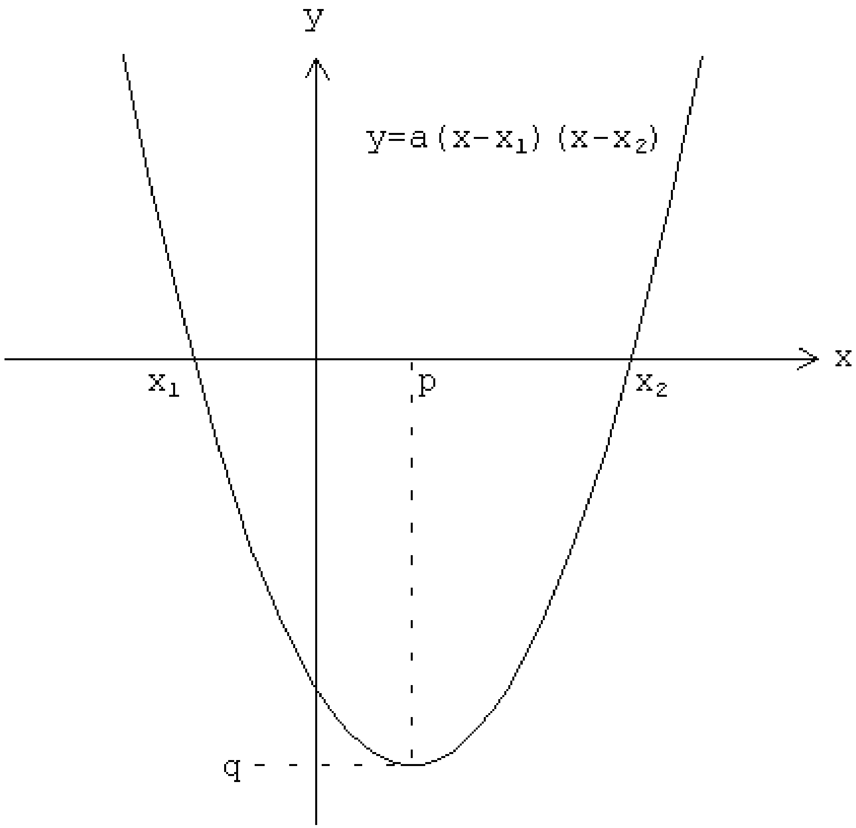
Examples of elementary functions

Polynomial



Examples of elementary functions

Quadratic function



$$y = ax^2 + bx + c$$

$$y = a(x - p)^2 + q$$

$$\Delta = b^2 - 4ac$$

$$x_{1/2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$p = -\frac{b}{2a}$$

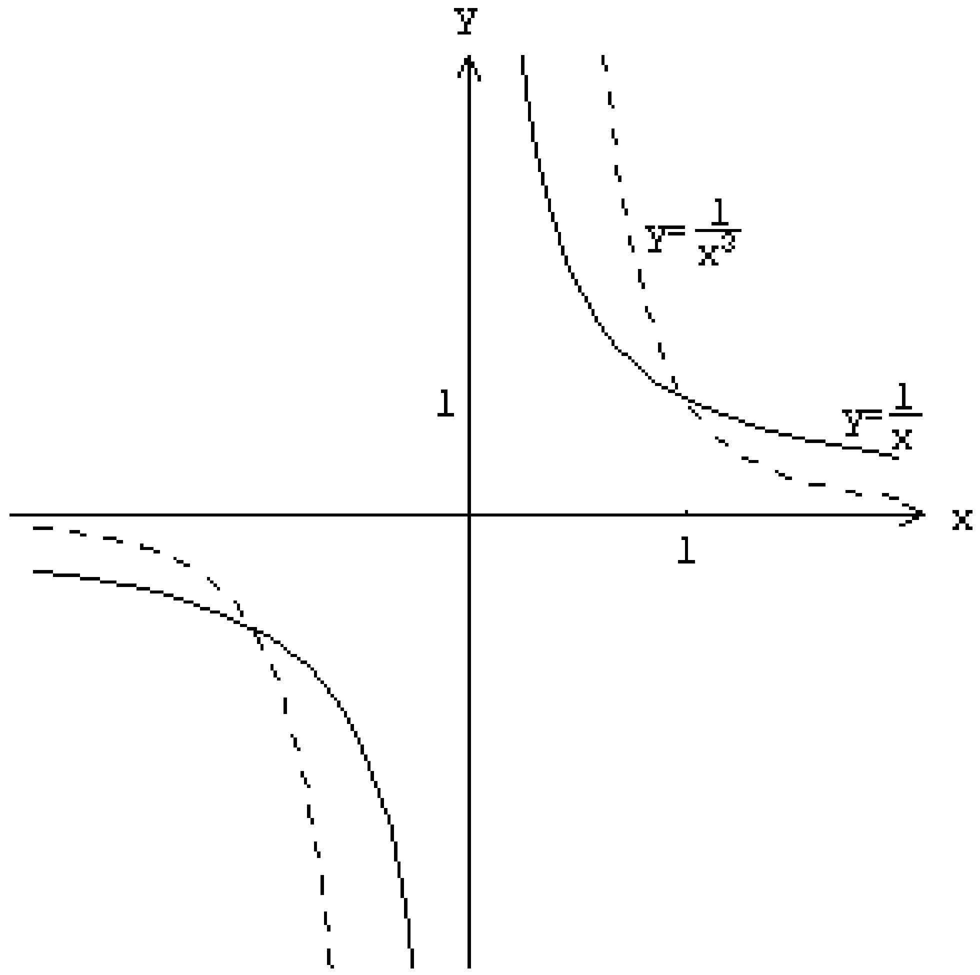
$$q = -\frac{\Delta}{4a}$$

Here $a > 0$ and $\Delta > 0$.

Examples of elementary functions

Power function

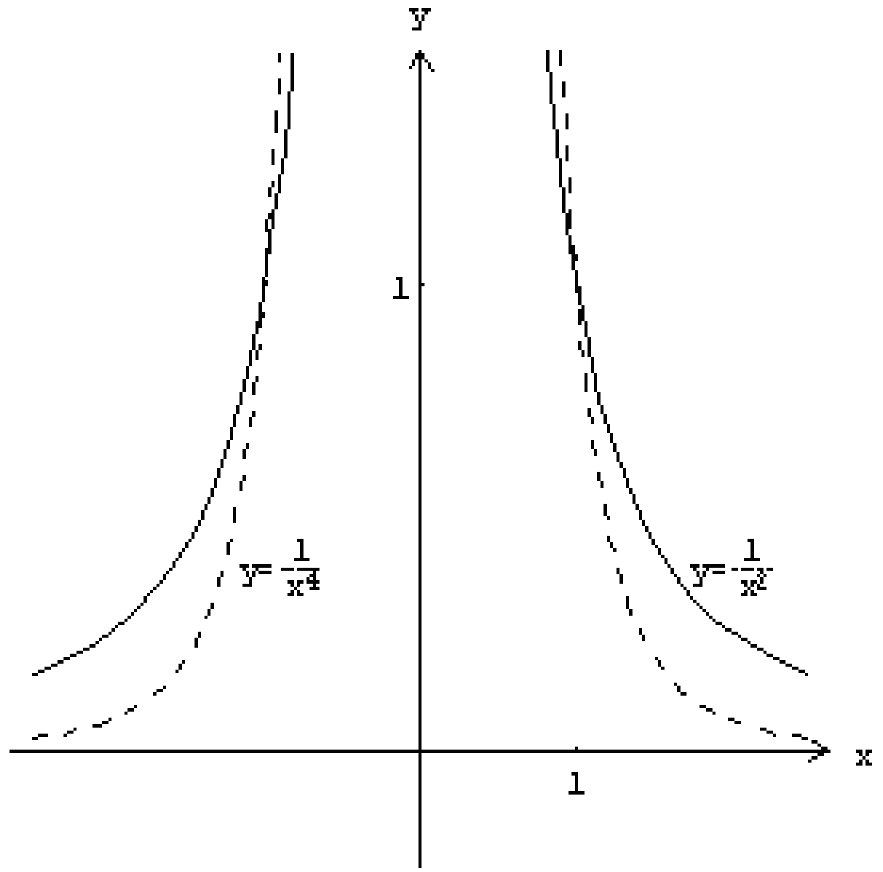
$$y = \frac{1}{x^n}, n \in \mathbb{N}, n \ - \text{odd}$$



Examples of elementary functions

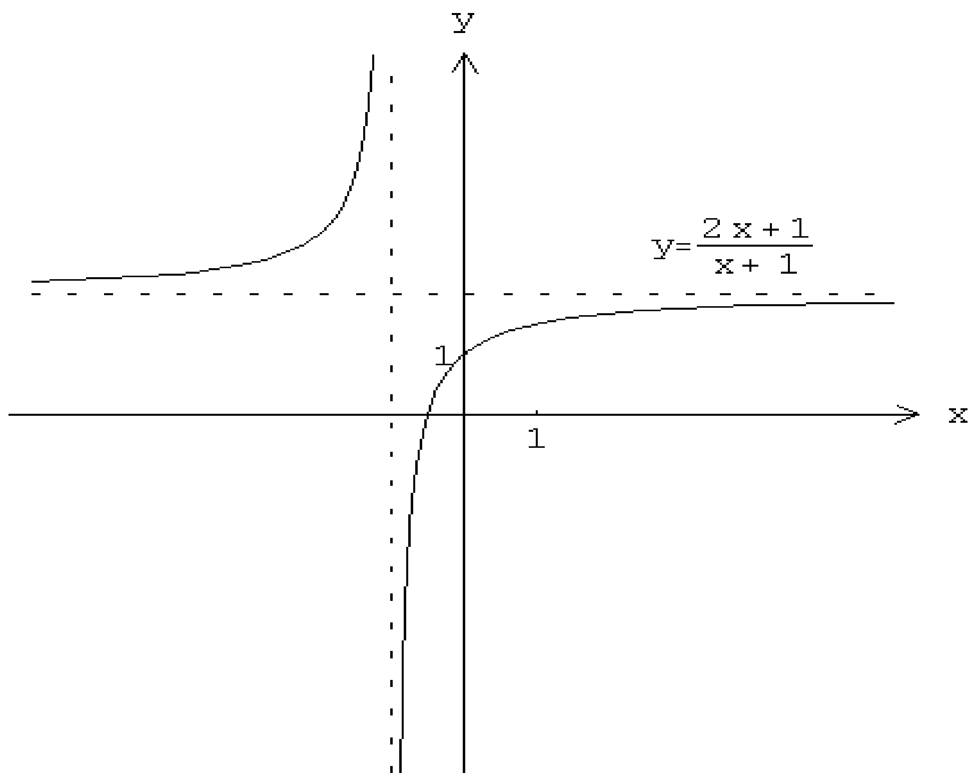
Power function

$$y = \frac{1}{x^n}, n \in \mathbb{N}, n \ -\text{even}$$



Examples of elementary functions

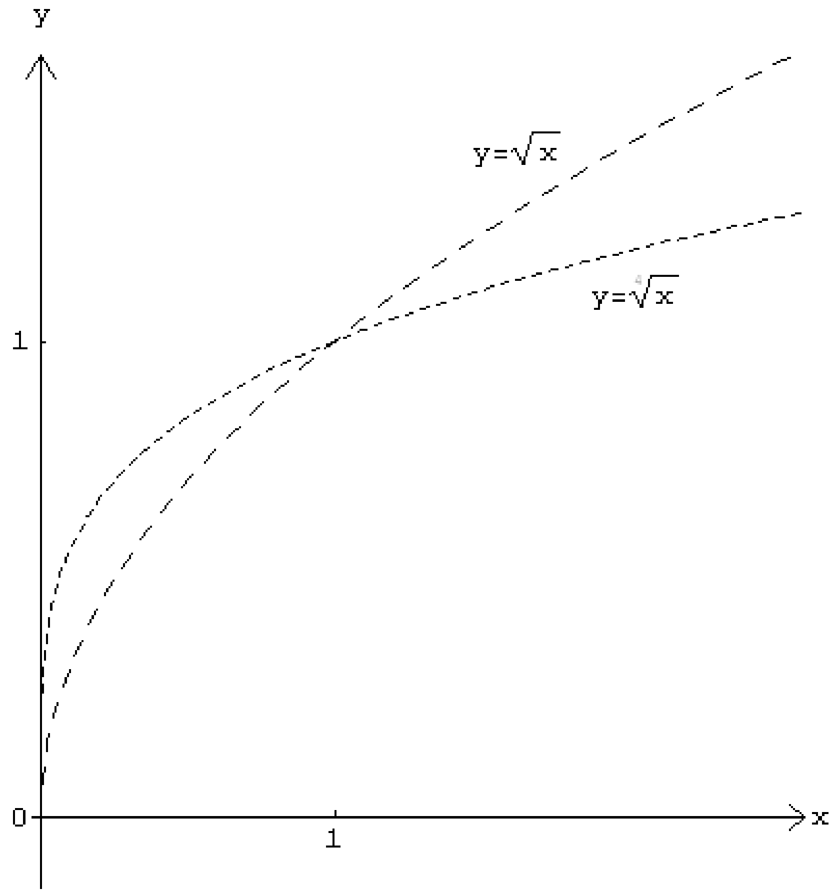
Homography $y = \dfrac{ax+b}{cx+d} \quad (ad \neq bc)$



Examples of elementary functions

Power function

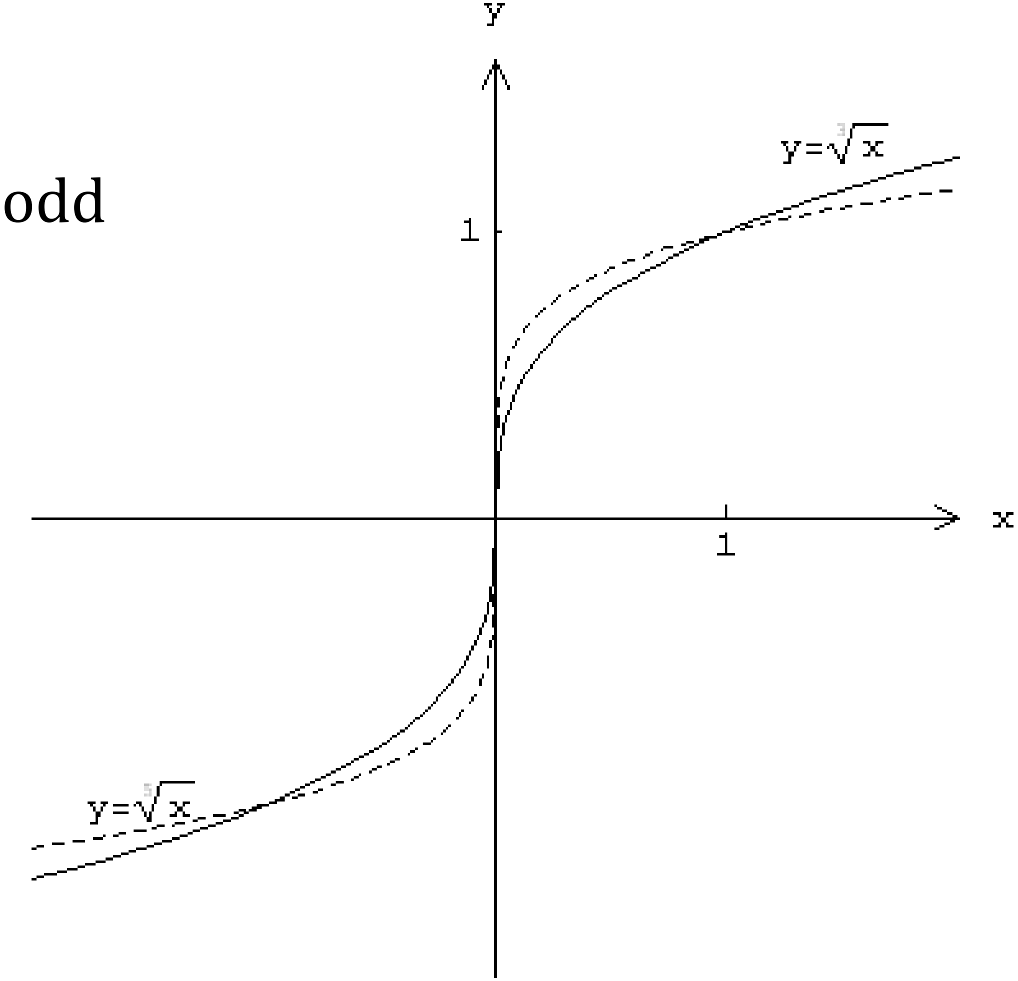
$y = \sqrt[n]{x}, n \in \mathbb{N}, n$ − even



Examples of elementary functions

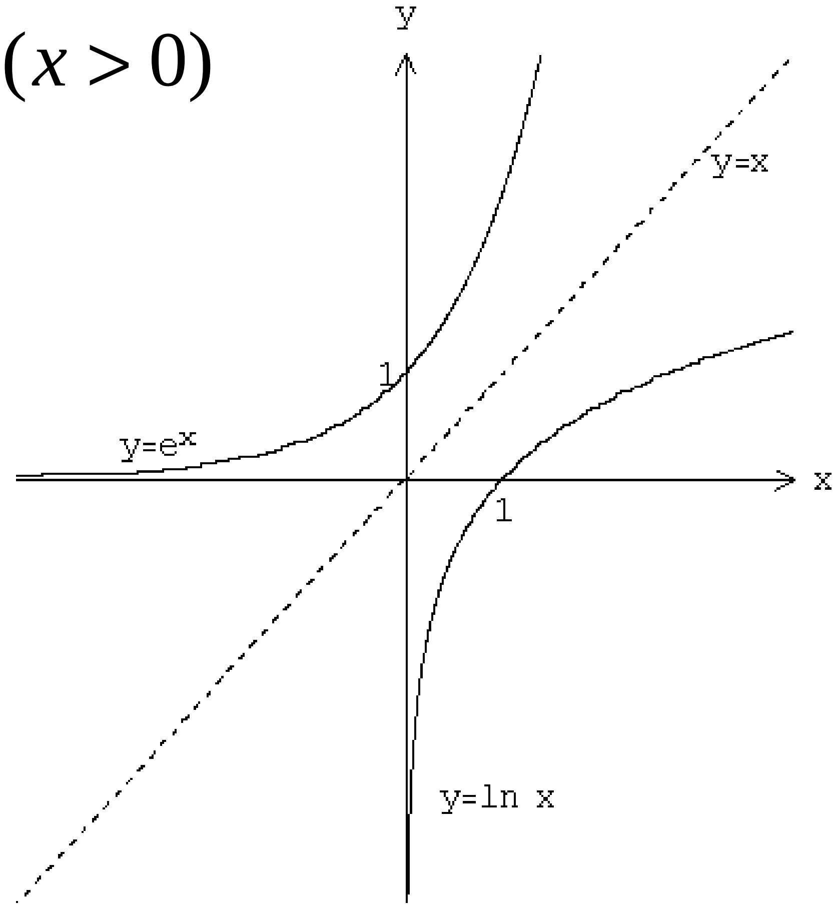
Power function

$y = \sqrt[n]{x}, n \in \mathbb{N}, n\ -$ odd



Examples of elementary functions

$$y = \ln x \Leftrightarrow e^y = x \; (x > 0)$$



Properties of logarithms

$$\log_a b = c \Leftrightarrow a^c = b \quad (a > 0,\ a \neq 1,\ b > 0)$$

$$\log_a b + \log_a c = \log_a (bc)$$

$$\log_a b - \log_a c = \log_a \frac{b}{c}$$

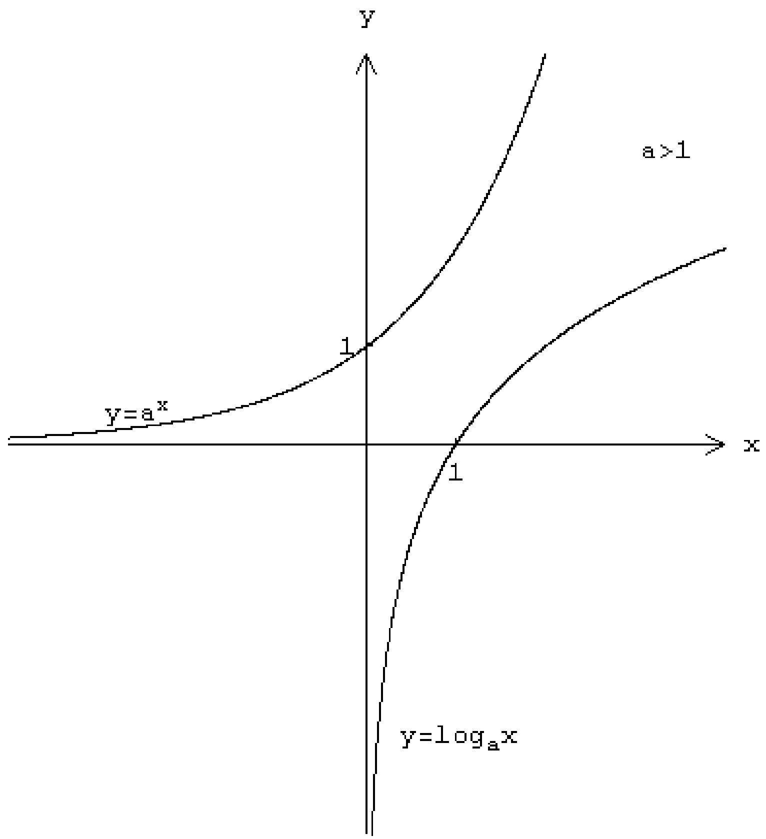
$$\log_a b^k = k \log_a b$$

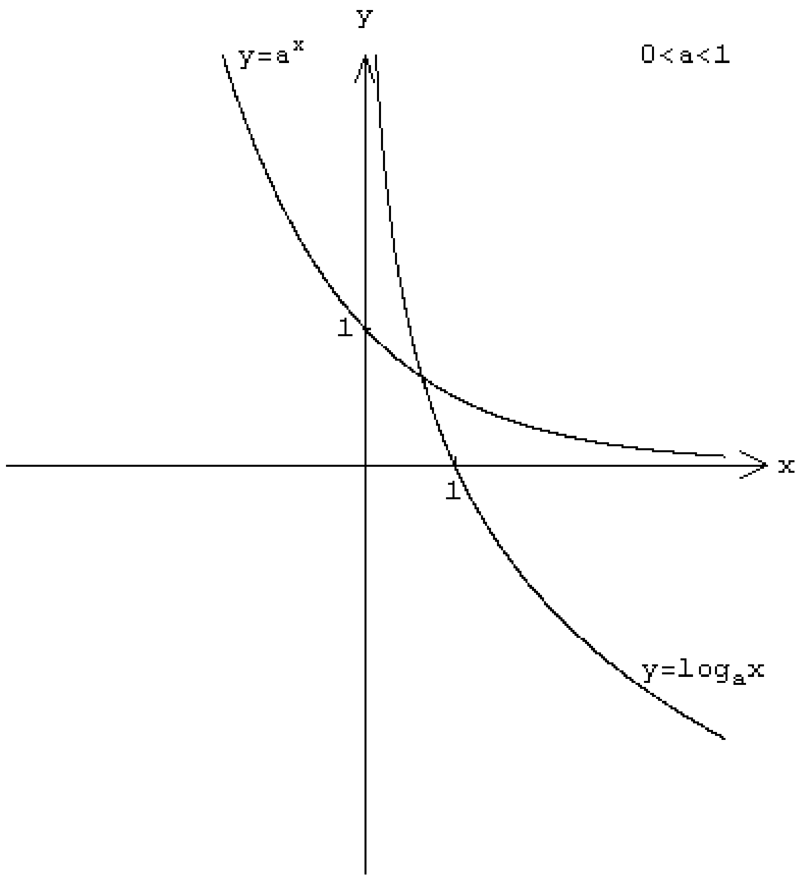
$$\log_a b = \frac{\log_c b}{\log_c a} = \frac{\ln b}{\ln a}$$

What we have to assume in formulae above?

Examples of elementary functions

Exponential and logarithmic functions

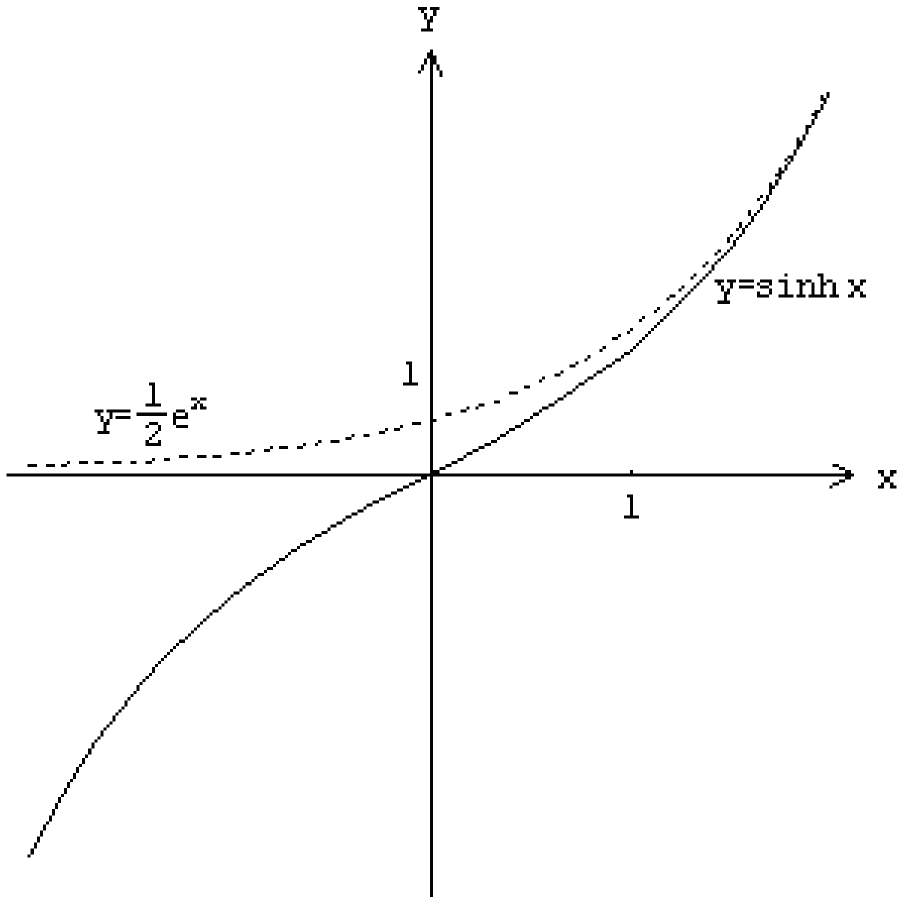




Examples of elementary functions

Hyperbolic functions

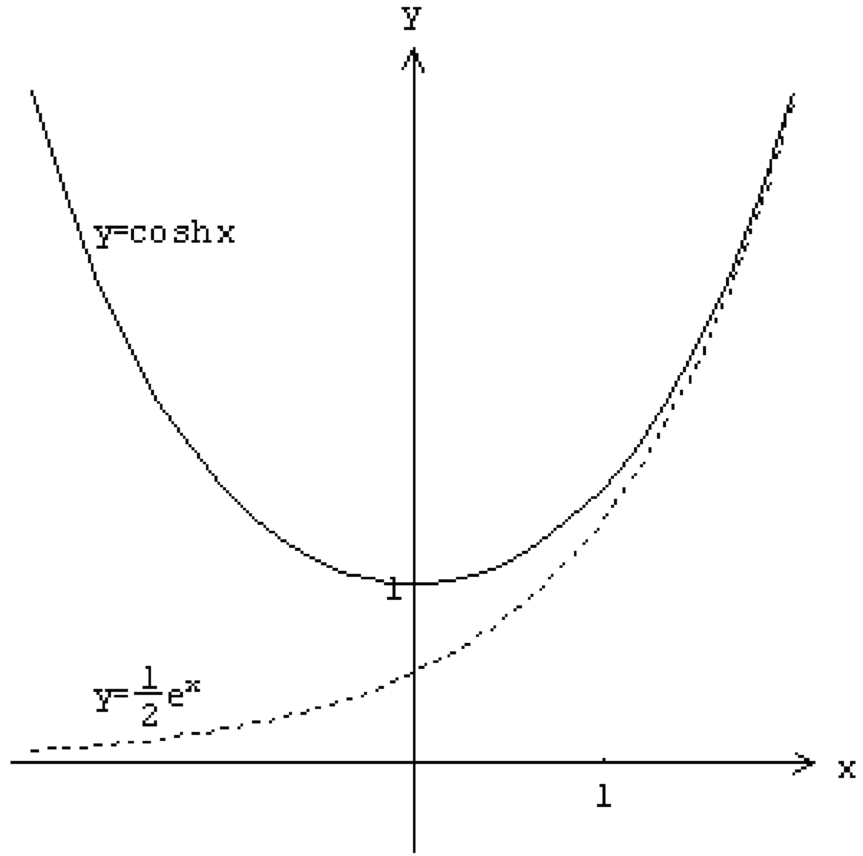
$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$



Examples of elementary functions

Hyperbolic functions

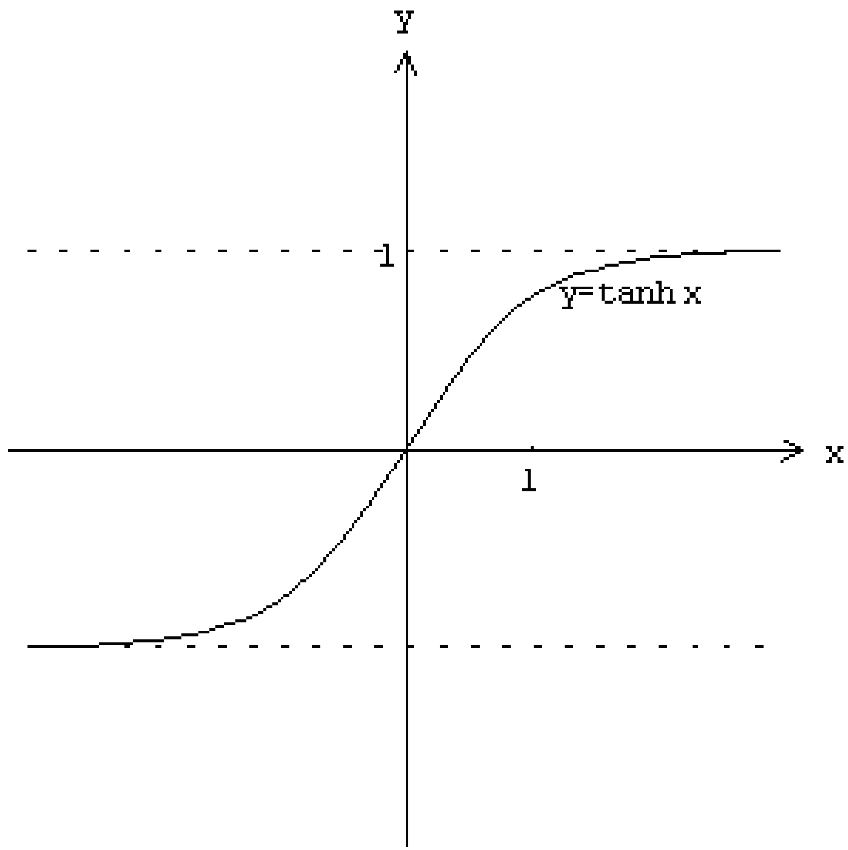
$$\cosh x = \frac{e^{x} + e^{-x}}{2}$$



Examples of elementary functions

Hyperbolic functions

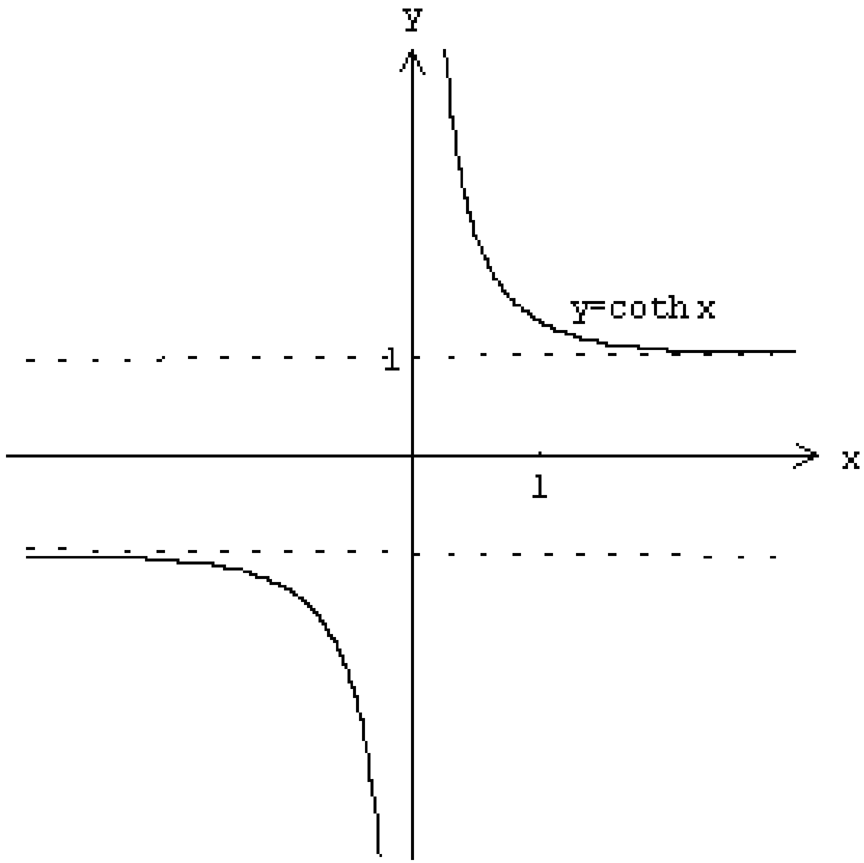
$$\tanh x = \frac{\sinh x}{\cosh x}$$



Examples of elementary functions

Hyperbolic functions

$$\coth x = \frac{1}{\tanh x}$$



Some properties of hyperbolic functions

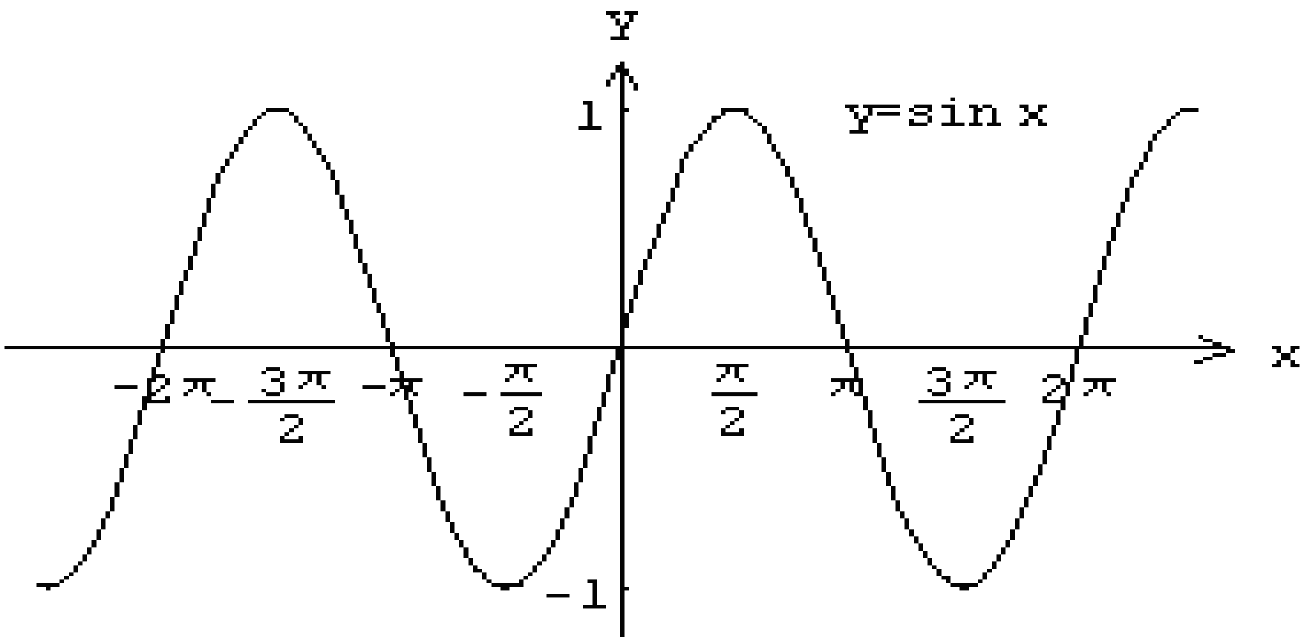
$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

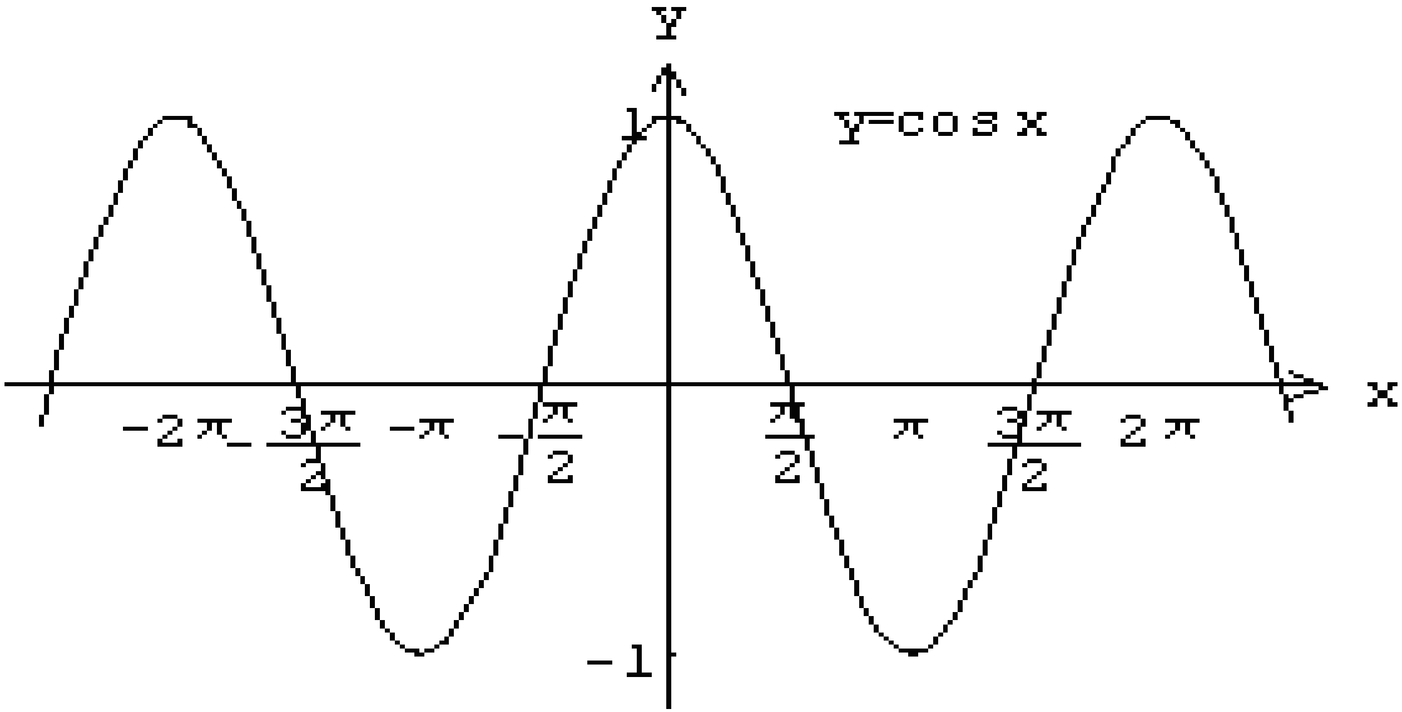
Examples of elementary functions

Trigonometric functions



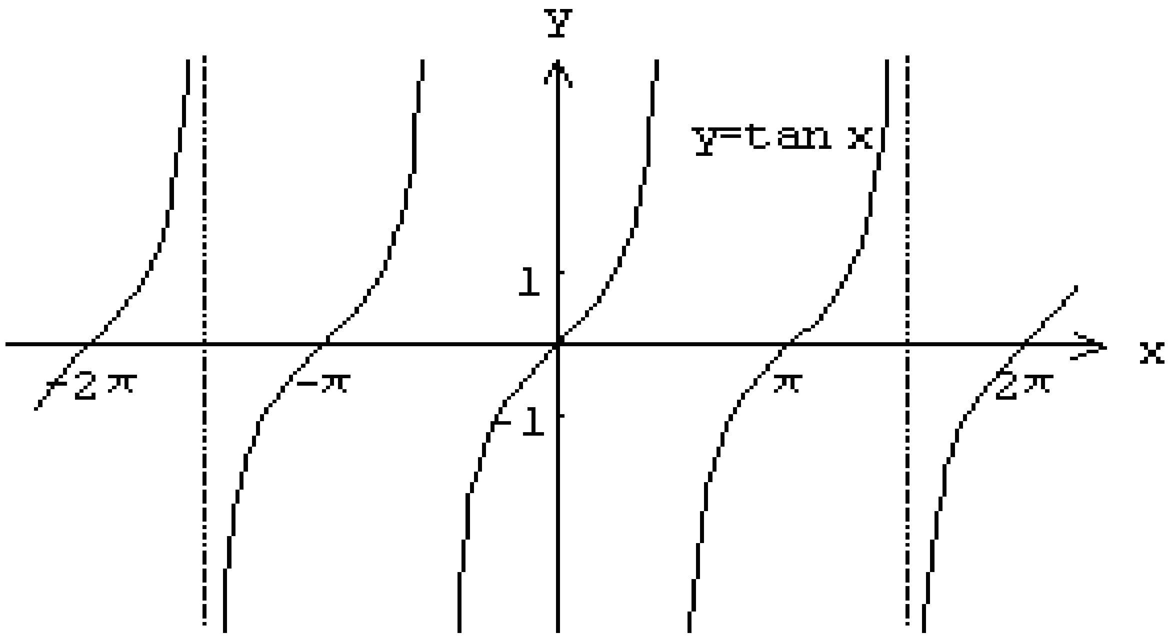
Examples of elementary functions

Trigonometric functions



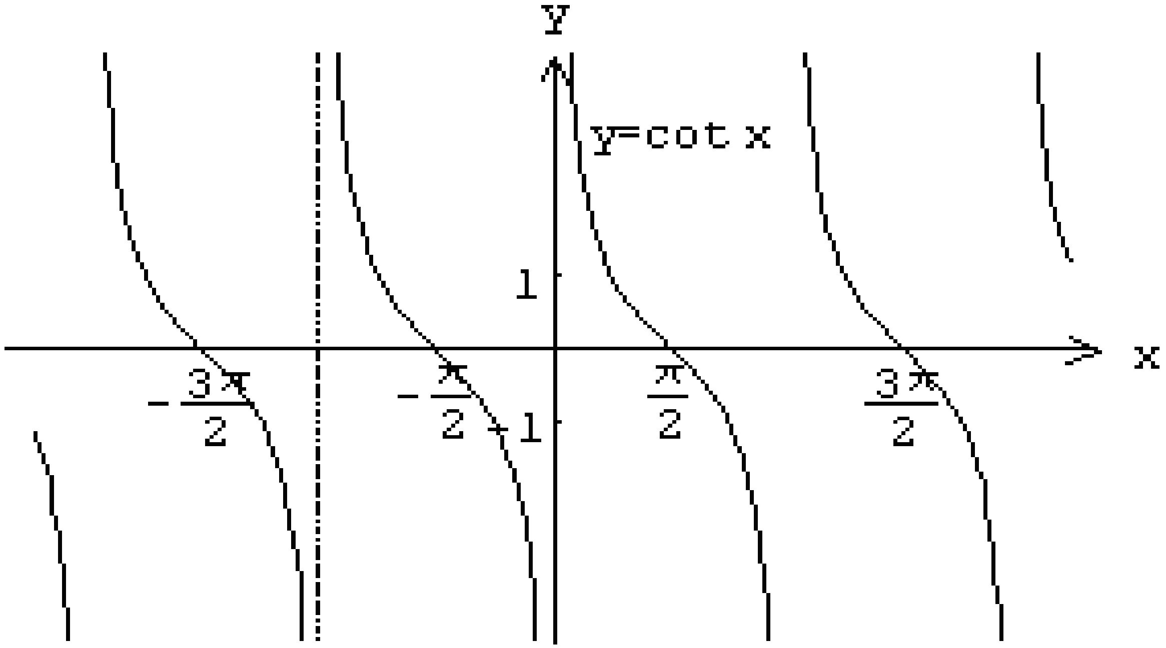
Examples of elementary functions

Trigonometric functions



Examples of elementary functions

Trigonometric functions



Some properties of trigonometric functions

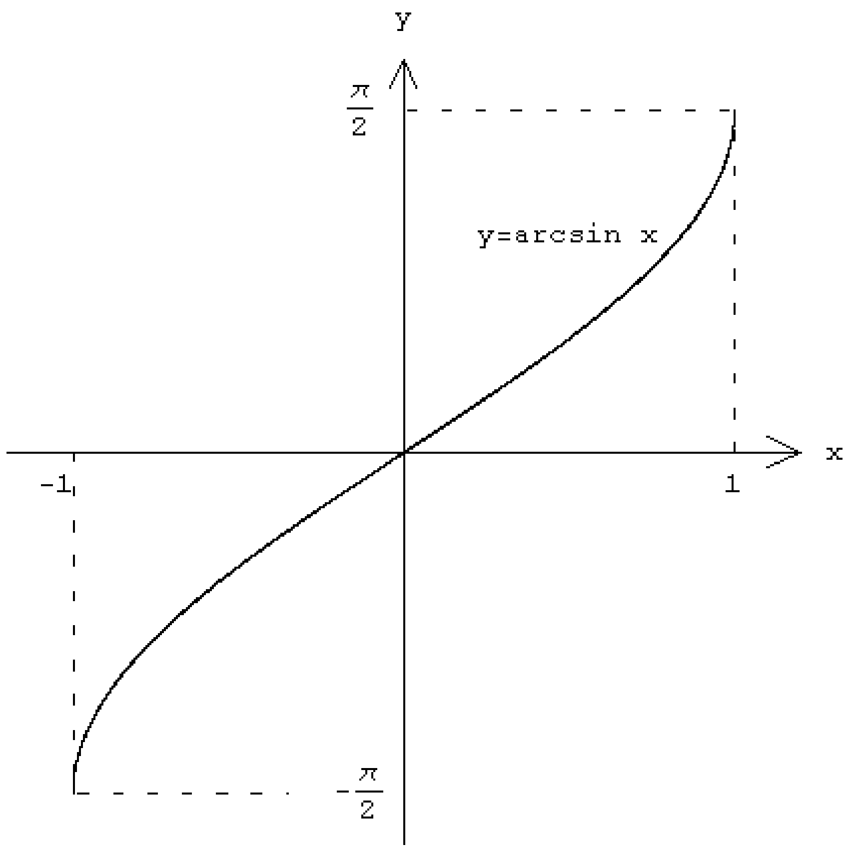
$$\sin^2 x + \cos^2 x = 1$$
$$\tan x \cot x = 1$$
$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x =$$
$$= 1 - 2\sin^2 x = 2\cos^2 x - 1$$
$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$
$$\sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2}$$
$$\cos x + \cos y = 2 \cos \frac{x + y}{2} \cos \frac{x - y}{2}$$
$$\cos x - \cos y = -2 \sin \frac{x + y}{2} \sin \frac{x - y}{2}$$

$$\sin(\pi - x) = \sin x$$
$$\sin(\pi + x) = -\sin x$$
$$\cos(\pi - x) = -\cos x$$
$$\cos(\pi + x) = -\cos x$$
$$\tan(\pi - x) = -\tan x$$
$$\sin\left(\frac{\pi}{2} + x\right) = \cos x$$
$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$$
$$\tan\left(\frac{\pi}{2} + x\right) = -\cot x$$
$$\cos\left(\frac{3\pi}{2} + x\right) = \sin x$$
$$\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$$

Examples of elementary functions

Inverse trigonometric functions

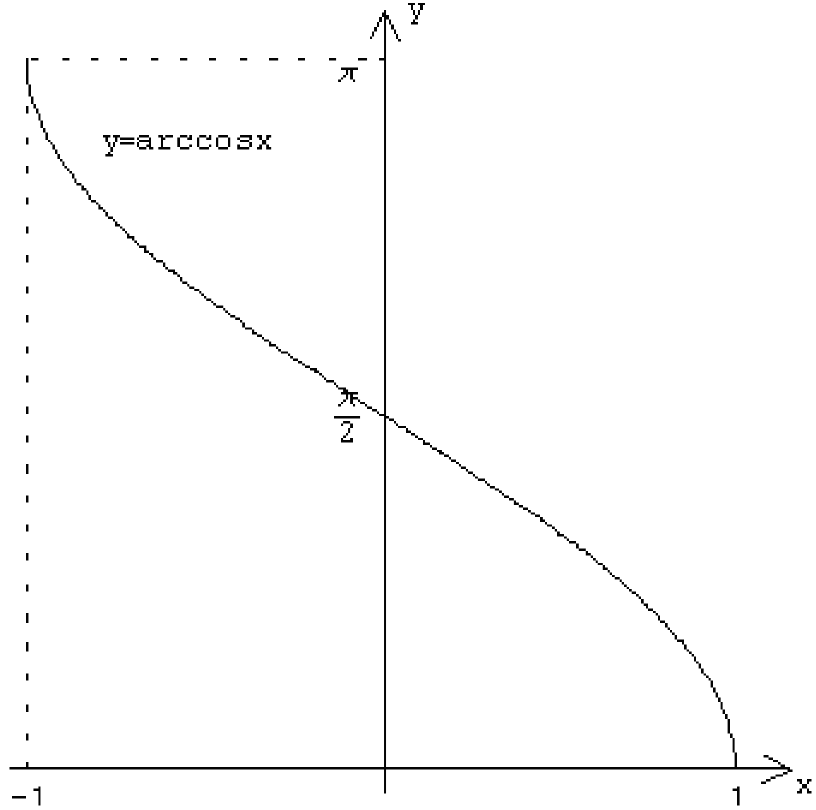
$$y = \arcsin x \Leftrightarrow x = \sin y\,,$$
$$x \in \langle -1,1 \rangle, y \in \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$$



Examples of elementary functions

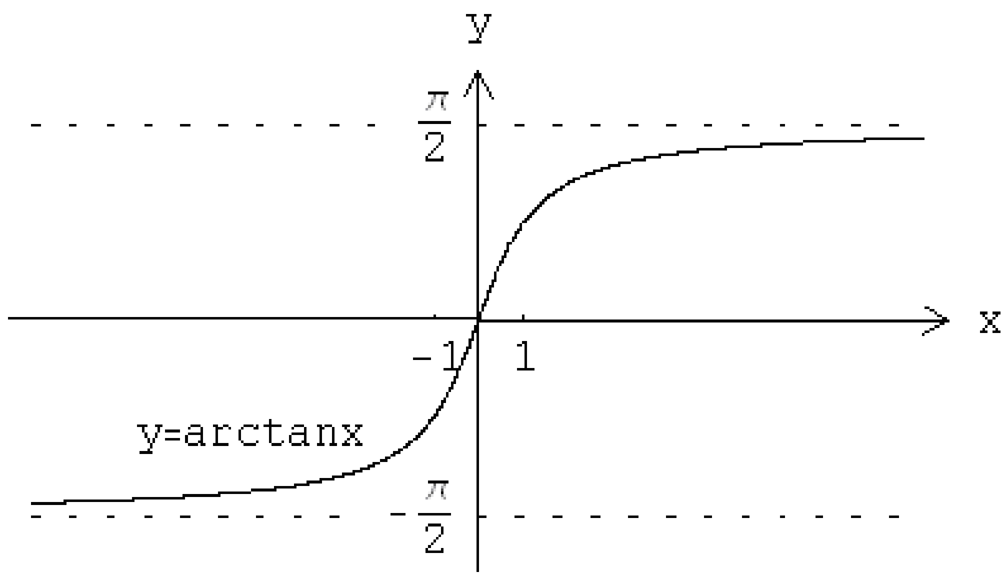
Inverse trigonometric functions

$$y = \arccos x \Leftrightarrow x = \cos y,$$
$$x \in \langle -1,1 \rangle, y \in \langle 0, \pi \rangle$$



Examples of elementary functions

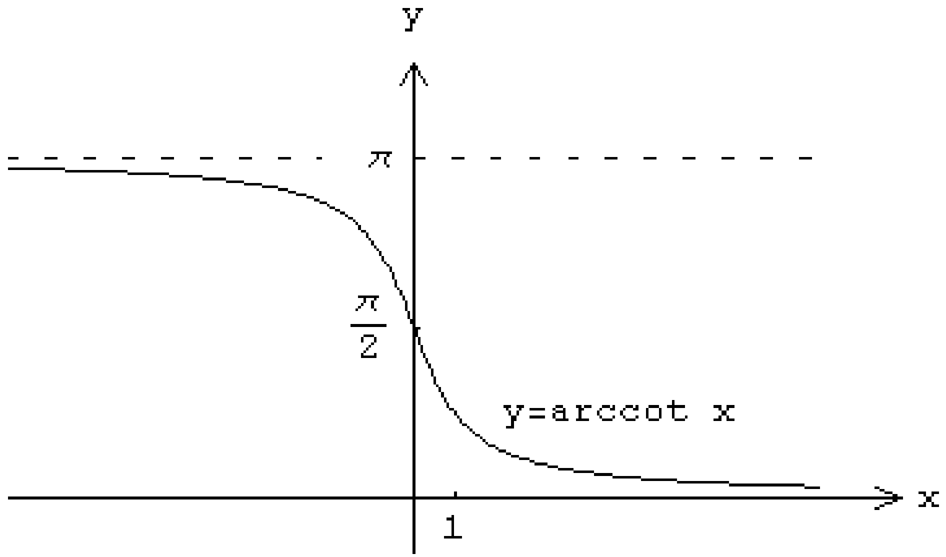
Inverse trigonometric functions



$$y = \arctan x \Leftrightarrow x = \tan y, \qquad x \in \mathbb{R}, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Examples of elementary functions

Inverse trigonometric functions



$$y = \operatorname{arccot} x \Leftrightarrow x = \cot y, \qquad x \in \mathbb{R}, y \in (0, \pi)$$

Example: calculate the exact value of $\cos(\operatorname{arccot}(-2))$.

$$x = \cos(\operatorname{arccot}(-2)) = \cos a = -\frac{2}{\sqrt{5}}$$

$$\operatorname{arccot}(-2) = a, \quad a \in \left(\frac{\pi}{2}, \pi\right)$$

$$\cot a = -2$$

$$\frac{\cos a}{\sin a} = -2$$

$$\cos a = -2\sin a$$

$$\cos^2 a = 4\sin^2 a$$

$$\cos^2 a = 4 - 4\cos^2 a$$

$$\cos^2 a = \frac{4}{5}$$

$$\cos a = \pm\frac{2}{\sqrt{5}}$$

Example. Show that:

1) $\arcsin x + \arccos x = \frac{\pi}{2}$ for $x \in \langle -1, 1\rangle$;

2) $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$ for $x \in \mathbb{R}$;

3) $\sin(\arcsin x) = x$ for $x \in \langle -1, 1\rangle$;

4) $\arcsin(\sin x) = ?$

Elementary functions of several variables

Let us consider a real-valued function of n real variables, where n is fixed, natural and greater than 1.

$$f: D \to \mathbb{R}, D \subset \mathbb{R}^n$$
$$y = f(x_1, x_2, \cdots, x_n)$$

The following functions (the projection onto the ith coordinate) are basic elementary functions:

$$\pi_i (x_1, x_2, \cdots, x_n) = x_i, \qquad i = 1,2, \cdots, n.$$

Elementary functions of several variables are functions obtained from projections and their compositions with elementary functions of one variable by operations listed in Df. 16.

Example:

The function $z(x, y) = 3x^2 + \ln(x - \sqrt{y})$ is elementary because

$$z(x, y) = 3(\pi_1(x, y))^2 + \ln(\pi_1(x, y) - \sqrt{\pi_2(x, y)})$$

constant multiple

product of two el. f-ns

difference of two el. f-ns

composition of square root with el. f-n

composition of ln *with el. f-n*

sum of two el. f-ns