PART 3
METRIC AND METRIC SPACES

Df. 1. Let X be a nonempty set. A metric (or distance function) on set X is a function $d: X \times X \to \mathbb{R}$ such that the following conditions are satisfied:

1) $\forall_{a,b \in X} \; d(a,b) = 0 \Leftrightarrow a = b$

2) $\forall_{a,b \in X} \; d(a,b) = d(b,a)$

3) $\forall_{a,b,c \in X} \; d(a,b) + d(b,c) \geq d(a,c)$.

Df. 2. An ordered pair (X, d), where d is a metric on nonempty set X, is called a metric space.

Examples:

1. A discrete metric on nonempty set X

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

2. $X = \mathbb{R},\ d(a,b) = |a - b|$

3. $X = \mathbb{C},\ d(z_1, z_2) = |z_1 - z_2|$

4. $X = \mathbb{R}^2,\ d(P(a,b), Q(c,d)) = \sqrt{(a-c)^2 + (b-d)^2}$

5. $X -$ set of real-valued functions of one real variable,

defined and bounded on $\langle 0,1 \rangle$

$$d(f,g) = \sup_{x \in \langle 0,1 \rangle} |f(x) - g(x)|$$

Note: metrics 2-4 are called Euclidean metrics.

Examples:

Non Euclidean metrics on $\mathbb{R}^2$:

$$P(a, b), Q(c, d)$$

1) taxi (NY) metric

$$d(P,Q) = |a-c| + |b-d|$$

2) river metric

$$d(P,Q) = \begin{cases} |b| + |d| + |a-c| & \text{if } a \neq c \\ |b-d| & \text{if } a = c \end{cases}$$

3) Rome (center) metric

$$d(P,Q) = \begin{cases} \sqrt{a^2+b^2} + \sqrt{c^2+d^2} & \text{if } \overrightarrow{OP} \nparallel \overrightarrow{OQ} \\ \sqrt{(a-c)^2 + (b-d)^2} & \text{if } \overrightarrow{OP} \parallel \overrightarrow{OQ} \end{cases}$$

Points and sets in metric spaces

Let $X \neq \emptyset$

(X, d) – metric space

$P, Q \in X$ – elements of X (called *points*)

$S \subset X$ – subset of X.

Df. 3. A complement of S is the set $S' = X \setminus S$.

Remark: $(S')' = S,\ \ X' = \emptyset,\ \ \emptyset' = X,\ \ S \cup S' = X,\ \ S \cap S' = \emptyset.$

Df. 4. A ball with center P and radius $\varepsilon > 0$ is the set

$$B(P, \varepsilon) = \{Q \in X\colon\ d(P, Q) < \varepsilon\}.$$

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Df. 5. A neighborhood of P is any ball with center P:

$$N_P = \mathrm{nbd}(P) = B(P, \varepsilon).$$

Df. 6. A deleted neighborhood of P is any ball with center P without the center:

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$$D_P = B(P, \varepsilon) \setminus \{P\}.$$

Df. 7. A set S is called bounded iff it is contained in some ball.

Df. 8. A point P is called the cluster point of S iff every deleted nbd of P contains at least one point of S.

Df. 9. A set S is called closed iff it contains all its cluster points.

Df. 10. The closure of set S is the smallest closed set containing S

$$\mathrm{cl}(S) = \{P \in X: P \in S \ \vee \ P \text{ is the cluster point of } S\}.$$

Df. 11. A point P is the interior point of S iff P has a nbd lying entirely in S. The set of all interior points is called the interior of S and it is denoted by $\mathrm{Int}(S)$

$$\mathrm{Int}(S) = \{P \in S\colon\ \exists_{\varepsilon>0}\, B(P,\varepsilon) \subset S\}.$$

Df. 12. A set S is called open iff every member of S has a nbd contained entirely in S, i.e.

$$\forall_{P\in S}\exists_{\varepsilon>0}\ \ B(P,\varepsilon) \subset S$$

or, equivalently, $S = \mathrm{Int}(S)$.

Df. 13. A point P from S is called the isolated point of S iff there exists a deleted nbd of P which contains no points of S, i.e.

$$\exists_{\varepsilon>0}\, B(P,\varepsilon) \cap S = \{P\}.$$

Df. 14. A point P is called the frontier point of S iff every nbd of P contains both points from S and points from S'. The set of all frontier points is called the frontier of S and it is denoted by $\mathrm{Fr}(S)$

$$\mathrm{Fr}(S) = \left\{P \in X\colon \exists_{\varepsilon>0}\exists_{Q_1\in S}\exists_{Q_2\notin S}\ \{Q_1, Q_2\} \subset B(P,\varepsilon)\right\}.$$

Note 1: the frontier of S is made up of points which are either

- members of S and cluster points of S'

or

- members of S' and cluster points of S.

Note 2: the frontier of S is the set of all

- isolated points of S

and

- cluster points of S which are not interior points of S.

Df. 15. The diameter of a nonempty set S is the least upper bound of all distances between any two points of S:

$$\operatorname{diam}(S) = \sup_{P,Q\in S} d(P,Q)\,.$$

Df. 16. A region is a nonempty open subset R of $\mathbf{R}^2$ whose any two points can be joined by a polygonal arc lying entirely in R.

Df. 17. A closed region $= \operatorname{cl}(R)$, where R is a region.