

PART 4  
SEQUENCES

# Definitions

**Df. 1.** An **infinite sequence** with terms from set  $X$  is any function of the form  $a: \mathbb{N} \rightarrow X$ .

**Df. 2.** A **finite sequence** with  $k$  terms ( $k$  – fixed natural number) from set  $X$  is any function of the form  $a: \{1, 2, 3, \dots, k\} \rightarrow X$ .

**We will study only infinite sequences.**

To determine the sequence it's sufficient to give the terms  $a(1), a(2), a(3), \dots$  in the proper order. For simplicity we write  $a(n) = a_n$  (here  $a_n$  denotes **the  $n$ th term**). Thus a sequence is

$$(a_1, a_2, a_3, \dots, a_n, \dots) = (a_n)_{n=1}^{\infty} = \{a_n\}_{n=1}^{\infty}.$$

**Note:** the notation with braces is traditional and it may be deceptive – remember that the order of terms is important!

# Limit of a sequence

**Df. 3.** Let  $(X, d)$  be a metric space and  $\{a_n\}$  be a sequence with terms from  $X$ .

$$\lim_{n \rightarrow +\infty} a_n = L \Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad d(a_n, L) < \varepsilon.$$

Notation:  $\lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow \infty} a_n = \lim_n a_n$

If the limit of sequence exists, then we say that the sequence is **convergent**. Otherwise we say that it is **divergent**.

- 1) In the Euclidean metric space  $(\mathbb{R}, d)$  we have  $d(a, b) = |a - b|$  so the definition of limit takes form:

$$\lim_n a_n = L \Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad |a_n - L| < \varepsilon.$$

$$a_n, b_n, a, b \in \mathbb{R}$$

2) In the Euclidean metric space  $(\mathbb{C}, d)$  we have  $d(z_1, z_2) = |z_1 - z_2|$  so and the definition of limit is:

$$\lim_n (a_n + ib_n) = a + ib \Leftrightarrow$$

$$\Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad |a_n + ib_n - a - ib| < \varepsilon$$

$$\Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \sqrt{(a_n - a)^2 + (b_n - b)^2} < \varepsilon$$

Hence  $\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad |a_n - a| < \varepsilon \wedge |b_n - b| < \varepsilon$

so

$$\lim_n (a_n + ib_n) = a + ib \Rightarrow \left[ \lim_n a_n = a \wedge \lim_n b_n = b \right]$$

$$a_n, b_n, a, b \in \mathbb{R}$$

Conversely,

$$\left[ \lim_n a_n = a \wedge \lim_n b_n = b \right] \Rightarrow \lim_n (a_n + ib_n) = a + ib$$

because

$$\forall \varepsilon > 0 \exists N_1 \in \mathbb{N} \forall n > N_1 \quad |a_n - a| < \frac{\varepsilon}{\sqrt{2}}, \quad \forall \varepsilon > 0 \exists N_2 \in \mathbb{N} \forall n > N_2 \quad |b_n - b| < \frac{\varepsilon}{\sqrt{2}}$$

$$N = \max\{N_1, N_2\}$$

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad |a_n + ib_n - a - ib| = \sqrt{(a_n - a)^2 + (b_n - b)^2} < \varepsilon$$

Therefore

$$\lim_n (a_n + ib_n) = a + ib \Leftrightarrow \left[ \lim_n a_n = a \wedge \lim_n b_n = b \right]$$

$$x_n, y_n, x_0, y_0 \in \mathbb{R}$$

3) In the Euclidean metric space  $(\mathbb{R}^2, d)$  we have

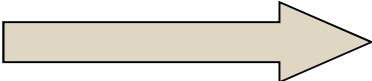
$$\begin{aligned} \lim_n P_n(x_n, y_n) = P_0(x_0, y_0) &\Leftrightarrow \\ &\Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad d(P_n, P_0) < \varepsilon \\ &\Leftrightarrow \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad \sqrt{(x_n - x_0)^2 + (y_n - y_0)^2} < \varepsilon \end{aligned}$$

So, analogously,

$$\lim_n P_n(x_n, y_n) = P_0(x_0, y_0) \Leftrightarrow \left[ \lim_n x_n = x_0 \wedge \lim_n y_n = y_0 \right]$$

**WHY?**

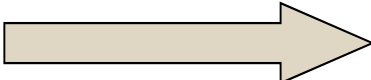
**Theorem 1.** The limit of a sequence is unique whenever the limit exists.

Proof  AFCC\*

\* B. Sikora, E. Łobos, *A First Course in Calculus*, Wydawnictwo Politechniki Śląskiej, Gliwice 2007

**Df. 4.** We say that  $\{a_n\}$  is **bounded** iff the set  $\{a_n : n \in \mathbb{N}\}$  is bounded.

**Theorem 2.** If a sequence is convergent, then it is bounded.

Proof  AFCC

Is the converse theorem true?

# Subsequences

1)  $\{a_n\} = (a_1, a_2, a_3, \dots)$

2)  $\{n_k\} = (n_1, n_2, n_3, \dots)$  such that

$$n_k \in \mathbb{N} \text{ and } n_1 < n_2 < n_3 < \dots$$

3)  $\{b_k\} = \{a_{n_k}\} = (a_{n_1}, a_{n_2}, a_{n_3}, \dots)$

subsequence of  $\{a_n\}$





# Important questions

Can we say something about a subsequence if we know that

- the sequence is convergent
- the sequence is divergent

Can we say something about a sequence if we know that

- all its subsequences are convergent to the same limit
- a subsequence is divergent
- there are two subsequences convergent to different limits

Can we say something about a sequence if we

- add finite number of terms
- discard finite number of terms
- add infinite number of terms
- discard infinite number of terms



# Reduction to real sequences

**Theorem 3.** Let  $\{a_n\}$  be a sequence with terms from  $X$ , and  $d$  be a metric on  $X$ . Then

$$\lim_n a_n = L \iff \lim_n d(a_n, L) = 0$$

where the limit on the right hand side is considered in the Euclidean metric space  $(\mathbb{R}, | \cdot |)$ .

**Proof:**

Let  $b_n = d(a_n, L)$ . Therefore  $b_n \geq 0$  and  $b_n \in \mathbb{R}$  so

$$\begin{aligned} RHS &\iff \lim_n b_n = 0 \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad d_E(b_n, 0) < \varepsilon \iff \\ &\iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad |b_n - 0| < \varepsilon \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad b_n < \varepsilon \iff \\ &\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \quad d(a_n, L) < \varepsilon \iff \lim_n a_n = L \iff LHS. \end{aligned}$$



# Limits of real sequences

*Now we consider sequences with real terms and Euclidean metric space  $(\mathbb{R}, | \cdot |)$ .*

**Df. 5.** (Improper limits)

$$\lim_n a_n = +\infty \iff \forall M \in \mathbb{R} \exists N \in \mathbb{N} \forall n > N \quad a_n > M$$

$$\lim_n a_n = -\infty \iff \forall m \in \mathbb{R} \exists N \in \mathbb{N} \forall n > N \quad a_n < m$$

**Df. 6.** (monotonic sequences)

$$\{a_n\} \text{ is increasing} \iff \forall n \in \mathbb{N} \quad a_{n+1} > a_n$$

$$\{a_n\} \text{ is decreasing} \iff \forall n \in \mathbb{N} \quad a_{n+1} < a_n$$

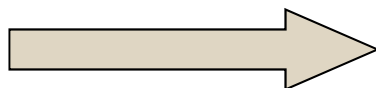
$$\{a_n\} \text{ is nonincreasing} \iff \forall n \in \mathbb{N} \quad a_{n+1} \leq a_n$$

$$\{a_n\} \text{ is nondecreasing} \iff \forall n \in \mathbb{N} \quad a_{n+1} \geq a_n$$

**Note:**  $\{a_n\}$  is bounded iff  $\exists M \in \mathbb{R} \forall n \in \mathbb{N} \quad a_n \leq M$ .

**Theorem 4.** Every bounded monotonic sequence is convergent.

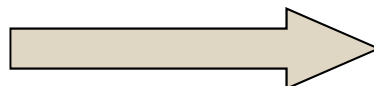
Proof



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**Theorem 5 (Squeeze Theorem).** If  $a_n \leq b_n \leq c_n$  for each natural  $n$ , and  $\lim_n a_n = \lim_n c_n = L$ , then  $\lim_n b_n = L$ .

Proof



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Example:  $\lim_{n \rightarrow \infty} \frac{\sin(\arctan n^3)}{n} = ?$

$$-1 \leq \sin(\arctan n^3) \leq 1$$

$$\frac{1}{n} \leq \frac{\sin(\arctan n^3)}{n} \leq \frac{1}{n}$$

$n \rightarrow \infty$

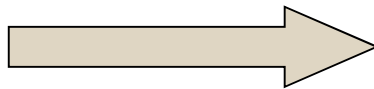
$$\lim_{n \rightarrow \infty} \frac{\sin(\arctan n^3)}{n} = 0.$$



**Theorem 6.** If  $\lim_n a_n = a$  and  $\lim_n b_n = b$ , then

- 1)  $\lim_n (a_n + b_n) = a + b$
- 2)  $\lim_n (ka_n) = ka \quad (k \in \mathbf{R})$
- 3)  $\lim_n (a_n \cdot b_n) = ab$
- 4)  $\lim_n \frac{a_n}{b_n} = \frac{a}{b} \quad (b_n \neq 0, b \neq 0)$
- 5)  $\lim_n |a_n| = |a|.$

Proof



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## Theorem 7.

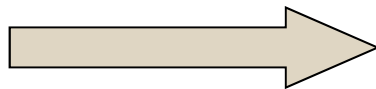
$$1) \lim_n |a_n| = 0 \Rightarrow \lim_n a_n = 0$$

$$2) \lim_n a_n = \pm\infty \Rightarrow \lim_n \frac{1}{a_n} = 0$$

$$3) \left[ \lim_n a_n = 0 \wedge \forall_n a_n > 0 \right] \Rightarrow \lim_n \frac{1}{a_n} = +\infty$$

$$4) \left[ \lim_n a_n = 0 \wedge \{b_n\} \text{ is bounded} \right] \Rightarrow \lim_n (a_n \cdot b_n) = 0$$

Proof



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- Which items in Theorems 6 and 7 hold for complex sequences?
- Be careful applying Theorem 6 – one has to be sure that the limits exist.
- Indeterminate forms:

$$\left[ \frac{0}{0} \right], \left[ \frac{\infty}{\infty} \right], [\infty - \infty], [0 \cdot \infty], [1^\infty], [\infty^0], [0^0]$$





## Examples:

$$1) \lim_{n \rightarrow \infty} (n^3 + 2n + 5) = +\infty$$

$$2) \lim_{n \rightarrow \infty} (n^2 - 3n + 2) = +\infty$$

$$3) \lim_{n \rightarrow \infty} (n^2 - 3n^5 + \sqrt{n}) = -\infty$$

$$4) \lim_{n \rightarrow \infty} \frac{2n + 3}{5 - 4n} = -\frac{1}{2}$$

$$5) \lim_{n \rightarrow \infty} \frac{2n^2 - 3n}{n^2 - 4n + 1} = 2$$

$$6) \lim_{n \rightarrow \infty} \frac{2n^2 - 3n}{3 - 4n} = -\infty$$

$$7) \lim_{n \rightarrow \infty} \frac{1 + 2n - 3n^3}{n - 2n^2} = +\infty$$

$$8) \lim_{n \rightarrow \infty} \frac{1 + 2n - 3n^2}{n^5 + 2n^2} = 0$$

$$9) \lim_{n \rightarrow \infty} \frac{3 - n + 3n^2}{n\sqrt{n} + 2n} = 0$$



# Important limits

$$1) \lim_n \sqrt[n]{n} = 1;$$

$$2) \lim_n \sqrt[n]{a} = 1 \quad (a > 0);$$

$$3) \lim_n \frac{a^n}{n!} = 0 \quad (a \in \mathbf{R});$$

$$4) \lim_n \left(1 + \frac{1}{n}\right)^n = e.$$

**DERIVE THESE LIMITS!**

**Theorem 8.**  $\lim_n b_n = \pm\infty \implies \lim_n \left(1 + \frac{1}{b_n}\right)^{b_n} = e.$

**Theorem 9.**  $\lim_n a_n = 0 \implies \lim_n (1 + a_n)^{\frac{1}{a_n}} = e.$

## Examples:

$$1) \lim_{n \rightarrow \infty} \left(1 + \frac{2}{3n}\right)^{4n} = [1^\infty]$$

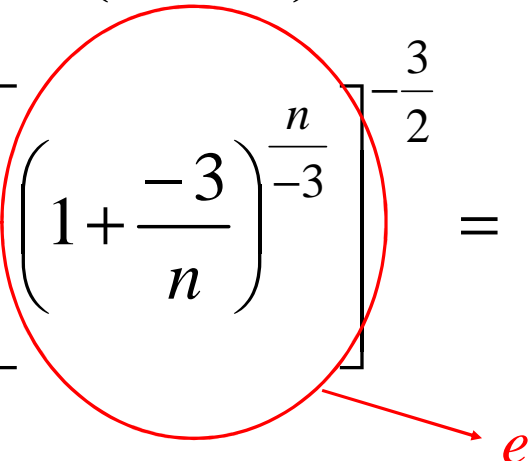
$$= \lim_{n \rightarrow \infty} \left(1 + \frac{2}{3n}\right)^{\frac{3n}{2} \cdot \frac{8}{3}} =$$

$$= \lim_{n \rightarrow \infty} \left[ \left(1 + \frac{2}{3n}\right)^{\frac{3n}{2}} \right]^{\frac{8}{3}} = e^{\frac{8}{3}} = \sqrt[3]{e^8} = e^2 \cdot \sqrt[3]{e^2}$$

$e$

$$2) \lim_{n \rightarrow \infty} \left(1 - \frac{3}{n}\right)^{\frac{n}{2}} = [1^\infty]$$

$$= \lim_{n \rightarrow \infty} \left(1 + \frac{-3}{n}\right)^{\frac{n}{-3} \cdot \left(-\frac{3}{2}\right)} =$$

$$= \lim_{n \rightarrow \infty} \left[ \left(1 + \frac{-3}{n}\right)^{\frac{n}{-3}} \right]^{-\frac{3}{2}} = e^{-\frac{3}{2}} = \frac{1}{e^{\frac{3}{2}}} = \frac{1}{e\sqrt{e}}$$


$$3) \lim_{n \rightarrow \infty} \left(\frac{n+1}{n+2}\right)^{-3n} = e^3$$

$$4) \lim_{n \rightarrow \infty} \left(\frac{2n-3}{2n+3}\right)^{2n} = e^{-6} = \frac{1}{e^6}$$

