PART 4

SEQUENCES

Definitions

Df. 1. An infinite sequence with terms from set X is any function of the form $a: \mathbb{N} \to X$.

Df. 2. A finite sequence with k terms (k – fixed natural number) from set X is any function of the form $a: \{1,2,3,\dots,k\} \to X$.

We will study only infinite sequences.

To determine the sequence it's sufficient to give the terms $a(1), a(2), a(3), \dots$ in the proper order. For simplicity we write $a(n) = a_n$ (here a_n denotes the nth term). Thus a sequence is

$$(a_1, a_2, a_3, \dots, a_n, \dots) = (a_n)_{n=1}^{\infty} = \{a_n\}_{n=1}^{\infty}.$$

Note: the notation with braces is traditional and it may be deceptive – remember that the order of terms is important!

Limit of a sequence

Df. 3. Let (X, d) be a metric space and $\{a_n\}$ be a sequence with terms from X.

$$\lim_{n\to+\infty} a_n = L \Leftrightarrow \forall_{\varepsilon>0}\exists_{N\in\mathbb{N}}\forall_{n>N} \ \ d(a_n, L) < \varepsilon.$$

Notation: $\lim_{n\to+\infty} a_n = \lim_{n\to\infty} a_n = \lim_{n} a_n$

If the limit of sequence exists, then we say that the sequence is convergent. Otherwise we say that it is divergent.

1) In the Euclidean metric space $(\mathbb{R}, d)$ we have $d(a, b) = |a - b|$ so the definition of limit takes form:

$$\lim_{n} a_n = L \Leftrightarrow \forall_{\varepsilon>0}\exists_{N\in\mathbb{N}}\forall_{n>N} \ \ |a_n - L| < \varepsilon.$$

$a_n, b_n, a, b \in \mathbb{R}$

2) In the Euclidean metric space $(\mathbb{C}, d)$ we have $d(z_1, z_2) = |z_1 - z_2|$ so and the definition of limit is:

$$\lim_n (a_n + ib_n) = a + ib \Leftrightarrow$$

$$\Leftrightarrow \forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n>N} \ \ |a_n + ib_n - a - ib| < \varepsilon$$

$$\Leftrightarrow \forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n>N} \sqrt{(a_n - a)^2 + (b_n - b)^2} < \varepsilon$$

Hence $\forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n>N} \ \ |a_n - a| < \varepsilon \ \ \wedge \ \ |b_n - b| < \varepsilon$

so

$$\lim_n (a_n + ib_n) = a + ib \Longrightarrow \left[\lim_n a_n = a \wedge \lim_n b_n = b\right]$$

$$a_n, b_n, a, b \in \mathbb{R}$$

Conversely,

$$\left[\lim_n a_n = a \wedge \lim_n b_n = b\right] \Longrightarrow \lim_n (a_n + ib_n) = a + ib$$

because

$$\forall_{\varepsilon>0} \exists_{N_1 \in \mathbb{N}} \forall_{n>N_1} \ |a_n - a| < \frac{\varepsilon}{\sqrt{2}}, \forall_{\varepsilon>0} \exists_{N_2 \in \mathbb{N}} \forall_{n>N_2} \ |b_n - b| < \frac{\varepsilon}{\sqrt{2}}$$

$$N = \max\{N_1, N_2\}$$

$$\forall_{\varepsilon>0} \exists_{N \in \mathbb{N}} \forall_{n>N} \ |a_n + ib_n - a - ib| = \sqrt{(a_n - a)^2 + (b_n - b)^2} < \varepsilon$$

Therefore

$$\lim_n (a_n + ib_n) = a + ib \Leftrightarrow \left[\lim_n a_n = a \wedge \lim_n b_n = b\right]$$

$$x_n, y_n, x_0, y_0 \in \mathbb{R}$$

3) In the Euclidean metric space $(\mathbb{R}^2, d)$ we have

$$\lim_n P_n(x_n, y_n) = P_0(x_0, y_0) \Leftrightarrow$$

$$\Leftrightarrow \forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n>N} \quad d(P_n, P_0) < \varepsilon$$

$$\Leftrightarrow \forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n>N} \quad \sqrt{(x_n - x_0)^2 + (y_n - y_0)^2} < \varepsilon$$

So, analogously,

$$\boxed{\lim_n P_n(x_n, y_n) = P_0(x_0, y_0) \Leftrightarrow \left[\lim_n x_n = x_0 \wedge \lim_n y_n = y_0\right]}$$

WHY?

Theorem 1. The limit of a sequence is unique whenever the limit exists.

Proof → AFCC*

* B. Sikora, E. Łobos, *A First Course in Calculus*, Wydawnictwo Politechniki Śląskiej, Gliwice 2007

Df. 4. We say that $\{a_n\}$ is bounded iff the set $\{a_n : n \in \mathbb{N}\}$ is bounded.

Theorem 2. If a sequence is convergent, then it is bounded.

Proof → AFCC

Is the coverse theorem true?

Subsequences

1) $\{a_n\} = (a_1, a_2, a_3, \ldots)$

2) $\{n_k\} = (n_1, n_2, n_3, \ldots)$ such that

$$n_k \in \mathbb{N} \text{ and } n_1 < n_2 < n_3 < \cdots$$

3) $\{b_k\} = \{a_{n_k}\} = \left(a_{n_1}, a_{n_2}, a_{n_3}, \ldots\right)$

subsequence of $\{a_n\}$

Important questions

Can we say something about a subsequence if we know that

- the sequence is convergent
- the sequence is divergent

Can we say something about a sequence if we know that

- all its subsequences are convergent to the same limit
- a subsequence is divergent
- there are two subsequences convergent to different limits

Can we say something about a sequence if we

- add finite number of terms
- discard finite number of terms
- add infinite number of terms
- discard infinite number of terms

Reduction to real sequences

Theorem 3. Let $\{a_n\}$be a sequence with terms from X, and d be a metric on X. Then

$$\lim_n a_n = L \Leftrightarrow \lim_n d(a_n, L) = 0$$

where the limit on the right hand side is considered in the Euclidean metric space $(\mathbb{R}, |\ \ |)$.

Proof:

Let $b_n = d(a_n, L)$. Therefore $b_n \geq 0$ and $b_n \in \mathbb{R}$ so

$$RHS \Leftrightarrow \lim_n b_n = 0 \Leftrightarrow \forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n>N} \ \ d_E(b_n, 0) < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n>N} \ \ |b_n - 0| < \varepsilon \Leftrightarrow \forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n>N} \ \ b_n < \varepsilon \Leftrightarrow$$

$$\forall_{\varepsilon>0} \exists_{N\in\mathbb{N}} \forall_{n>N} \ \ d(a_n, L) < \varepsilon \Leftrightarrow \lim_n a_n = L \Leftrightarrow LHS.$$

Limits of real sequences

Now we consider sequences with real terms and Euclidean metric space $(\mathbb{R}, |\ \ |)$.

Df. 5. (Improper limits)

$$\lim_n a_n = +\infty \Leftrightarrow \forall_{M\in\mathbb{R}} \exists_{N\in\mathbb{N}} \forall_{n>N} \quad a_n > M$$

$$\lim_n a_n = -\infty \Leftrightarrow \forall_{m\in\mathbb{R}} \exists_{N\in\mathbb{N}} \forall_{n>N} \quad a_n < m$$

Df. 6. (monotonic sequences)

$\{a_n\}$ is increasing $\Leftrightarrow \forall_{n\in\mathbb{N}} \quad a_{n+1} > a_n$

$\{a_n\}$ is decreasing $\Leftrightarrow \forall_{n\in\mathbb{N}} \quad a_{n+1} < a_n$

$\{a_n\}$ is nonincreasing $\Leftrightarrow \forall_{n\in\mathbb{N}} \quad a_{n+1} \leq a_n$

$\{a_n\}$ is nondecreasing $\Leftrightarrow \forall_{n\in\mathbb{N}} \quad a_{n+1} \geq a_n$

Note: $\{a_n\}$ is bounded iff $\exists_{M\in\mathbb{R}} \forall_{n\in\mathbb{N}} \quad a_n \leq M$.

Theorem 4. Every bounded monotonic sequence is convergent.

Proof $\Longrightarrow$ AFCC

Theorem 5 (Squeeze Theorem). If $a_n \leq b_n \leq c_n$ for each natural n, and $\lim\limits_{n} a_n = \lim\limits_{n} c_n = L$, then $\lim\limits_{n} b_n = L$.

Proof $\Longrightarrow$ AFCC

Example: $$\lim_{n\to\infty}\frac{\sin(\arctan n^3)}{n}=?$$

$$-1\le \sin(\arctan n^3)\le 1$$

$$-\frac{1}{n}\le \frac{\sin(\arctan n^3)}{n}\le \frac{1}{n}$$

$$n\to\infty$$

$$0$$

$$\lim_{n\to\infty}\frac{\sin(\arctan n^3)}{n}=0.$$

Theorem 6. If $\lim_n a_n = a$ and $\lim_n b_n = b$, then

1) $\lim_n (a_n + b_n) = a + b$
2) $\lim_n (k a_n) = ka \quad (k \in \mathbf{R})$
3) $\lim_n (a_n \cdot b_n) = ab$
4) $\lim_n \dfrac{a_n}{b_n} = \dfrac{a}{b} \quad (b_n \neq 0, b \neq 0)$
5) $\lim_n |a_n| = |a|$.

Proof → AFCC

Theorem 7.

1) $\lim\limits_{n} |a_n| = 0 \Rightarrow \lim\limits_{n} a_n = 0$

2) $\lim\limits_{n} a_n = \pm\infty \Rightarrow \lim\limits_{n} \dfrac{1}{a_n} = 0$

3) $\left[\lim\limits_{n} a_n = 0 \wedge \forall_n \; a_n > 0\right] \Rightarrow \lim\limits_{n} \dfrac{1}{a_n} = +\infty$

4) $\left[\lim\limits_{n} a_n = 0 \wedge \{b_n\} \text{ is bounded}\right] \Rightarrow \lim\limits_{n} (a_n \cdot b_n) = 0$

Proof $\Longrightarrow$ AFCC

• Which items in Theorems 6 and 7 hold for complex sequences?

• Be careful applying Theorem 6 – one has to be sure that the limits exist.

• Indeterminate forms:

$$\left[\frac{0}{0}\right],\ \left[\frac{\infty}{\infty}\right],\ [\infty-\infty],\ [0\cdot\infty],\ \left[1^{\infty}\right],\ \left[\infty^{0}\right],\ \left[0^{0}\right].$$

Examples:

1) $\lim_{n\to\infty}\left(n^3+2n+5\right)=+\infty$

2) $\lim_{n\to\infty}\left(n^2-3n+2\right)=+\infty$

3) $\lim_{n\to\infty}\left(n^2-3n^5+\sqrt{n}\right)=-\infty$

4) $\lim_{n\to\infty}\frac{2n+3}{5-4n}=-\frac{1}{2}$

5) $\lim_{n\to\infty}\frac{2n^2-3n}{n^2-4n+1}=2$

6) $\lim_{n\to\infty}\frac{2n^2-3n}{3-4n}=-\infty$

7) $\lim_{n\to\infty}\frac{1+2n-3n^3}{n-2n^2}=+\infty$

8) $\lim_{n\to\infty}\frac{1+2n-3n^2}{n^5+2n^2}=0$

9) $\lim_{n\to\infty}\frac{3-n+3n^2}{n\sqrt{n}+2n}=0$

Important limits

1) $\lim_n \sqrt[n]{n} = 1;$

2) $\lim_n \sqrt[n]{a} = 1 \ \ (a > 0);$

3) $\lim_n \frac{a^n}{n!} = 0 \ \ (a \in \mathbf{R});$

4) $\lim_n \left(1 + \frac{1}{n}\right)^n = e.$

DERIVE THESE LIMITS!

Theorem 8. $\lim_n b_n = \pm\infty \Longrightarrow \lim_n \left(1 + \frac{1}{b_n}\right)^{b_n} = e.$

Theorem 9. $\lim_n a_n = 0 \Longrightarrow \lim_n (1 + a_n)^{\frac{1}{a_n}} = e.$

Examples:

$$1)\ \lim_{n\to\infty}\left(1+\frac{2}{3n}\right)^{4n} = [1^{\infty}]$$

$$= \lim_{n\to\infty}\left(1+\frac{2}{3n}\right)^{\frac{3n}{2}\cdot\frac{8}{3}} =$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{2}{3n}\right)^{\frac{3n}{2}}\right]^{\frac{8}{3}} = e^{\frac{8}{3}} = \sqrt[3]{e^{8}} = e^{2}\cdot\sqrt[3]{e^{2}}$$

e

$$2)\ \lim_{n\to\infty}\left(1-\frac{3}{n}\right)^{\frac{n}{2}} = [1^{\infty}]$$

$$= \lim_{n\to\infty}\left(1+\frac{-3}{n}\right)^{\frac{n}{-3}\cdot\left(-\frac{3}{2}\right)} =$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{-3}{n}\right)^{\frac{n}{-3}}\right]^{-\frac{3}{2}} = e^{-\frac{3}{2}} = \frac{1}{e^{\frac{3}{2}}} = \frac{1}{e\sqrt{e}}$$

e

$$3)\ \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^{-3n} = e^{3}$$

$$4)\ \lim_{n\to\infty}\left(\frac{2n-3}{2n+3}\right)^{2n} = e^{-6} = \frac{1}{e^{6}}$$