PART 5

LIMITS OF FUNCTIONS AND CONTINUITY

The definition of limit of function *f* at point *a*

$$f: D \to Y,\ D \subset X,\ a - \text{cluster point of } D$$

$$(X, d_X),\ (Y, d_Y) - \text{metric spaces}$$

$$\lim_{x \to a} f(x) = L \Leftrightarrow \forall_{\substack{\{x_n\} \\ x_n \in D \\ x_n \neq a}} \left[\lim_n x_n = a \Rightarrow \lim_n f(x_n) = L \right]$$ Heine

possible convergence in different metric spaces

$$\lim_{x \to a} f(x) = L \Leftrightarrow$$ Cauchy

$$\Leftrightarrow \forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x \in D} \left[0 < d_X(x, a) < \delta \Rightarrow d_Y(f(x), L) < \varepsilon \right]$$

possible different metrics

Theorem 1. The Cauchy and Heine definitions of limit of function at a point are equivalent.

Proof → AFCC

Examples:

1) $$\lim_{x\to 2}\frac{1-x}{2x+1}=-\frac{1}{5}$$

2) $$\lim_{x\to 0}\frac{|x|}{x}$$ does not exist

3) $$\lim_{(x,y)\to(1,2)}\frac{x+y}{x-y}=-3$$

4) $$\lim_{(x,y)\to(1,1)}\frac{x-1}{1-y}$$ does not exist

The Heine definitions of one-sided limits

$f : D \to Y, D \subset \mathbb{R}, \quad a -$ cluster point of D,

$(\mathbb{R}, |\,|), (Y, d_Y)$ – metric spaces

$$\lim_{x \to a^+} f(x) = L \Leftrightarrow \forall_{\substack{\{x_n\} \\ x_n \in D \\ x_n > a}} \left[\lim_n x_n = a \Rightarrow \lim_n f(x_n) = L \right]$$

$$\lim_{x \to a^-} f(x) = L \Leftrightarrow \forall_{\substack{\{x_n\} \\ x_n \in D \\ x_n < a}} \left[\lim_n x_n = a \Rightarrow \lim_n f(x_n) = L \right]$$

The Cauchy definitions of one-sided limits

$f: D \to Y, D \subset \mathbb{R}, \quad a -$ cluster point of D,

$(\mathbb{R}, |\,|), (Y, d_Y)$ – metric spaces

$$\lim_{x \to a^+} f(x) = L \Leftrightarrow$$

$$\Leftrightarrow \forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x \in D} \left[a < x < a + \delta \Rightarrow d_Y(f(x), L) < \varepsilon \right]$$

$$\lim_{x \to a^-} f(x) = L \Leftrightarrow$$

$$\Leftrightarrow \forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x \in D} \left[a - \delta < x < a \Rightarrow d_Y(f(x), L) < \varepsilon \right]$$

Theorem 2. If $a \in \mathrm{Int}(D)$ then

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$$

Note 1: if $D = (a,b)$, then $\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x)$.

Note 2: Th. 2 may be generalized. How?

The Heine definition of improper limits

$f: D \to \mathbb{R}, D \subset X, \quad a -$ cluster point of D,

$(\mathbb{R}, |\ |), (X, d_X)$ – metric spaces

$$\lim_{x\to a} f(x) = +\infty \Leftrightarrow \underset{\substack{\{x_n\} \\ x_n \in D \\ x_n \neq a}}{\forall} \left[\lim_n x_n = a \Rightarrow \lim_n f(x_n) = +\infty \right]$$

$$\lim_{x\to a} f(x) = -\infty \Leftrightarrow \underset{\substack{\{x_n\} \\ x_n \in D \\ x_n \neq a}}{\forall} \left[\lim_n x_n = a \Rightarrow \lim_n f(x_n) = -\infty \right]$$

The Cauchy definition of improper limits

$f: D \to \mathbb{R}, D \subset X,\ \ a -$ cluster point of D,

$(\mathbb{R}, |\ |), (X, d_X)$ – metric spaces

$$\lim_{x \to a} f(x) = +\infty \Leftrightarrow$$
$$\Leftrightarrow \forall_M \exists_{\delta>0} \forall_{x \in D} \left[0 < d_X(x,a) < \delta \Rightarrow f(x) > M \right]$$

$$\lim_{x \to a} f(x) = -\infty \Leftrightarrow$$
$$\Leftrightarrow \forall_m \exists_{\delta>0} \forall_{x \in D} \left[0 < d_X(x,a) < \delta \Rightarrow f(x) < m \right]$$

The Heine definition of one-sided improper limits (1)

$f : D \to \mathbb{R}, D \subset \mathbb{R}, \quad a -$ cluster point of D,

$(\mathbb{R}, |\ |)$ – metric space

$$\lim_{x \to a^+} f(x) = +\infty \Leftrightarrow \underset{\substack{\{x_n\} \\ x_n \in D \\ x_n > a}}{\forall} \left[\lim_n x_n = a \Rightarrow \lim_n f(x_n) = +\infty \right]$$

$$\lim_{x \to a-} f(x) = +\infty \Leftrightarrow \underset{\substack{\{x_n\} \\ x_n \in D \\ x_n < a}}{\forall} \left[\lim_n x_n = a \Rightarrow \lim_n f(x_n) = +\infty \right]$$

The Heine definition of one-sided improper limits (2)

$f : D \to \mathbb{R}, D \subset \mathbb{R},\ a$ – cluster point of D,

$(\mathbb{R}, |\ |)$ – metric space

$$\lim_{x \to a^+} f(x) = -\infty \Leftrightarrow \underset{\substack{\{x_n\} \\ x_n \in D \\ x_n > a}}{\forall} \left[\lim_n x_n = a \Rightarrow \lim_n f(x_n) = -\infty \right]$$

$$\lim_{x \to a^-} f(x) = -\infty \Leftrightarrow \underset{\substack{\{x_n\} \\ x_n \in D \\ x_n < a}}{\forall} \left[\lim_n x_n = a \Rightarrow \lim_n f(x_n) = -\infty \right]$$

The Cauchy definition of one-sided improper limits (1)

$f : D \to \mathbb{R}, D \subset \mathbb{R}, \quad a -$ cluster point of D,

$(\mathbb{R}, |\ |)$ – metric space

$$\lim_{x \to a^+} f(x) = +\infty \Leftrightarrow$$

$$\Leftrightarrow \forall_M \exists_{\delta>0} \forall_{x \in D} \left[a < x < a + \delta \Rightarrow f(x) > M \right]$$

$$\lim_{x \to a^-} f(x) = +\infty \Leftrightarrow$$

$$\Leftrightarrow \forall_M \exists_{\delta>0} \forall_{x \in D} \left[a - \delta < x < a \Rightarrow f(x) > M \right]$$

The Cauchy definition of one-sided improper limits (2)

$f: D \to \mathbb{R}, D \subset \mathbb{R}, \quad a -$ cluster point of D,

$(\mathbb{R}, |\;|)$ – metric space

$$\lim_{x \to a^+} f(x) = -\infty \Leftrightarrow$$

$$\Leftrightarrow \forall_m \exists_{\delta > 0} \forall_{x \in D} \left[a < x < a + \delta \Rightarrow f(x) < m \right]$$

$$\lim_{x \to a^-} f(x) = -\infty \Leftrightarrow$$

$$\Leftrightarrow \forall_m \exists_{\delta > 0} \forall_{x \in D} \left[a - \delta < x < a \Rightarrow f(x) < m \right]$$

The Heine definition of limit at infinity

$f: D \to Y, D \subset \mathbb{R},\ D -$ unbounded above,

$(\mathbb{R}, |\ |), (Y, d_Y)$– metric spaces

$$\lim_{x \to +\infty} f(x) = L \Leftrightarrow \underset{\substack{\{x_n\} \\ x_n \in D}}{\forall} \left[\lim_n x_n = +\infty \Rightarrow \lim_n f(x_n) = L \right]$$

$f: D \to Y, D \subset \mathbb{R},\ D -$ unbounded below,

$(\mathbb{R}, |\ |), (Y, d_Y)$– metric spaces

$$\lim_{x \to -\infty} f(x) = L \Leftrightarrow \underset{\substack{\{x_n\} \\ x_n \in D}}{\forall} \left[\lim_n x_n = -\infty \Rightarrow \lim_n f(x_n) = L \right]$$

The Cauchy definition of limit at infinity

$f : D \to Y, D \subset \mathbb{R},\ D -$ unbounded above,

$(\mathbb{R}, |\ |), (Y, d_Y)$– metric spaces

$$\lim_{x \to +\infty} f(x) = L \Leftrightarrow$$

$$\Leftrightarrow \forall_{\varepsilon>0} \exists_{\delta} \forall_{x \in D} \left[x > \delta \Rightarrow d_Y(f(x), L) < \varepsilon \right]$$

$f : D \to Y, D \subset \mathbb{R},\ D -$ unbounded below,

$(\mathbb{R}, |\ |), (Y, d_Y)$– metric spaces

$$\lim_{x \to -\infty} f(x) = L \Leftrightarrow$$

$$\Leftrightarrow \forall_{\varepsilon>0} \exists_{\delta} \forall_{x \in D} \left[x < \delta \Rightarrow d_Y(f(x), L) < \varepsilon \right]$$

The Heine definitions of improper limits at infinity (1)

$f : D \to \mathbb{R}, D \subset \mathbb{R},\ D -$ unbounded above,

$(\mathbb{R}, |\ |)$ – metric space

$$\lim_{x\to+\infty} f(x) = +\infty \Leftrightarrow \forall_{\substack{\{x_n\} \\ x_n\in D}} \left[\lim_n x_n = +\infty \Rightarrow \lim_n f(x_n) = +\infty\right]$$

$$\lim_{x\to+\infty} f(x) = -\infty \Leftrightarrow \forall_{\substack{\{x_n\} \\ x_n\in D}} \left[\lim_n x_n = +\infty \Rightarrow \lim_n f(x_n) = -\infty\right]$$

The Heine definitions of improper limits at infinity (2)

$f: D \to \mathbb{R}, D \subset \mathbb{R},\ \ D -$ unbounded below,

$(\mathbb{R}, |\ |)$ – metric space

$$\lim_{x \to -\infty} f(x) = +\infty \Leftrightarrow \underset{x_n \in D}{\underset{\{x_n\}}{\forall}} \left[\lim_n x_n = -\infty \Rightarrow \lim_n f(x_n) = +\infty \right]$$

$$\lim_{x \to -\infty} f(x) = -\infty \Leftrightarrow \underset{x_n \in D}{\underset{\{x_n\}}{\forall}} \left[\lim_n x_n = -\infty \Rightarrow \lim_n f(x_n) = -\infty \right]$$

The Cauchy definitions of improper limits at infinity (1)

$f: D \to \mathbb{R}, D \subset \mathbb{R},\ D -$ unbounded above,

$(\mathbb{R}, |\ |)$ – metric space

$$\lim_{x \to +\infty} f(x) = +\infty \Leftrightarrow \forall_M \exists_\delta \forall_{x \in D} \left[x > \delta \Rightarrow f(x) > M \right]$$

$$\lim_{x \to +\infty} f(x) = -\infty \Leftrightarrow \forall_m \exists_\delta \forall_{x \in D} \left[x > \delta \Rightarrow f(x) < m \right]$$

The Cauchy definitions of improper limits at infinity (2)

$f: D \to \mathbb{R}, D \subset \mathbb{R},\ D -$ unbounded below,

$(\mathbb{R}, |\ |)$ – metric space

$$\lim_{x \to -\infty} f(x) = +\infty \Leftrightarrow \forall_M \exists_\delta \forall_{x \in D} \left[x < \delta \Rightarrow f(x) > M \right]$$

$$\lim_{x \to -\infty} f(x) = -\infty \Leftrightarrow \forall_m \exists_\delta \forall_{x \in D} \left[x < \delta \Rightarrow f(x) < m \right]$$

Theorem 3. If f and g are real (or complex) functions, a is the cluster point of domains of both functions, and

$$\lim_{x\to a} f(x) = F, \lim_{x\to a} g(x) = G, \text{ then:}$$

1) $\lim\limits_{x\to a}[f(x)+g(x)] = F+G$

2) $\lim\limits_{x\to a}[kf(x)] = kF \quad (k \in \mathbb{R})$

3) $\lim\limits_{x\to a}[f(x)\cdot g(x)] = FG$

4) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{F}{G} \quad (g(x)\neq 0, G\neq 0).$

Proof: Apply the Heine definition and Th.6/Part 4. ■

Theorem 4. If

- f , g, h are real-valued functions,
- a is the cluster point of domains of all functions,
- in some nbd of a, $f(x) \le g(x) \le h(x)$,
- $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

then $\lim_{x \to a} g(x) = L.$

Proof: Apply the Heine definition and Th.5/Part 4. ■

Important limits

1) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

2) $\lim_{x\to +\infty} \left(1+\frac{1}{x}\right)^x = e$

3) $\lim_{x\to -\infty} \left(1+\frac{1}{x}\right)^x = e$

4) $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

5) $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a \quad (a > 0)$

CAN YOU DERIVE THESE LIMITS?

Continuity

- f is continuous at a point
- f is continuous on a set
- f is continuous

The definition of function continuous at a point

$$f : D \to Y,\ D \subset X,\ a \in D$$

$(X, d_X), (Y, d_Y)$ – metric spaces

Function f is continuous at point a iff

$$\forall_{\substack{\{x_n\} \\ x_n \in D}} \left[\lim_n x_n = a \Rightarrow \lim_n f(x_n) = f(a)\right]$$

or, equivalently,

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x \in D} \left[d_X(x,a) < \delta \Rightarrow d_Y(f(x), f(a)) < \varepsilon\right].$$

WHEN A FUNCTION IS CONTINUOUS AT A POINT?

$a \in D$

a is the cluster point of D

a is the isolated point of D

We say that f is continuous on set A (where A is a subset of domain of f) iff f is continuous at each point from A.

We say that f is continuous iff f is continuous at each point of its domain.

Discontinuities

If *a* belongs to the domain of f and f is not continuous at a, the we say that f is discontinuous at a.

Note: if *a* is not the member of domain, then we do not define continuity at this point.

Points of discontinuity (or discontinuities) :

• members of domain at which function is discontinuous

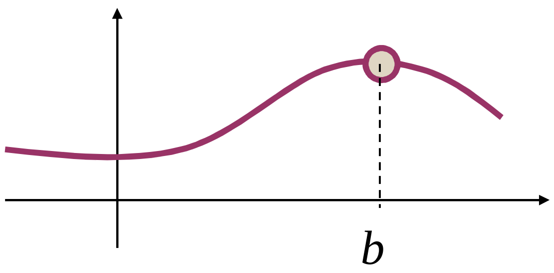
or

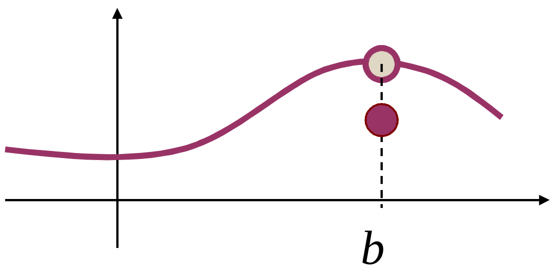
• cluster points of domain which are not members of domain

Classification of discontinuities of real-valued functions of one real variable

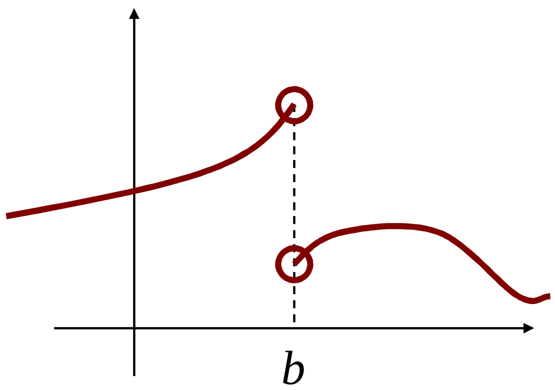
I type

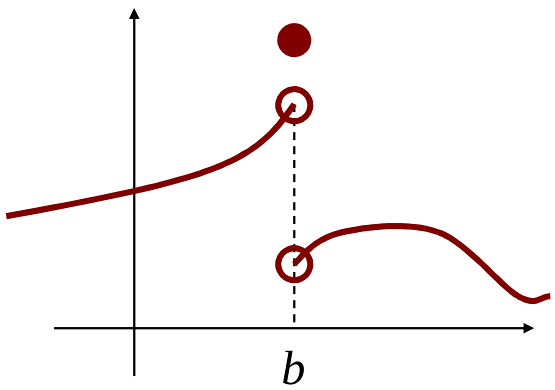
a) removable

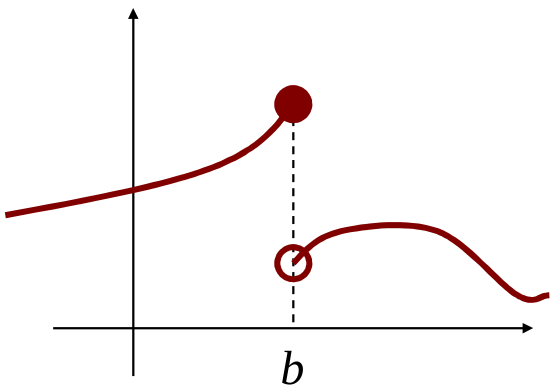




b) finite jump



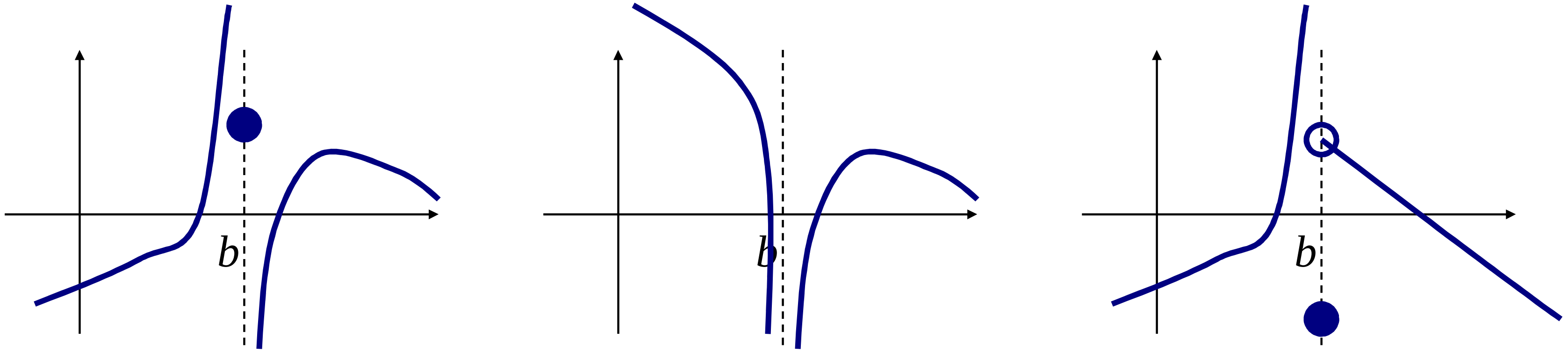




Classification of discontinuities of real-valued functions of one real variable

II type

a) infinite jump



b) oscillating discontinuity as $f(x) = \sin\frac{1}{x}$ at 0

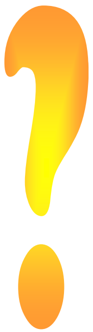
c) other

Find an example of a function $f : \mathbb{R} \to \mathbb{R}$

- that is continuous at exactly one point;

- such that f^2 is continuous but f is discontinuous at each real number;

- that is continuous at each rational number and discontinuous at each irrational number.



Properties of continuous functions

Theorem 5. If f and g are real-valued functions both continuous at a, then functions

- kf (k – real constant)
- $f+g$
- fg
- $\dfrac{f}{g}$ (where $g(a) \neq 0$)

are continuous at a.

Proof: Apply the Heine definition and Th.6/Part 4. ∎

Theorem 6. If f is continuous at a and g is continuous at $b = f(a)$, then $g \circ f$ is continuous at a.

Proof:

$$\lim_{x \to a} f(x) = f(a) = b$$

$$\lim_{y \to b} g(y) = g(b) = g(f(a))$$

$$h(x) = (g \circ f)(x) = g(f(x))$$

$$\lim_{x \to a} h(x) = \lim_{x \to a} g(f(x)) = \left| \begin{matrix} f(x) = y \\ x \to a \Longrightarrow y \to b \end{matrix} \right| = \lim_{y \to b} g(y) = g(b) =$$

$$= g\big(f(a)\big) = h(a)$$

■

The continuity of inverse function is not an easy matter.

Theorem 7. The inverse of a strictly monotonic continuous function is continuous in the interval where it is defined.

Show that Th. 7 gives sufficient but not necessary condition for continuity of inverse function.

Theorem 8. Each basic elementary function is continuous.

Proof  AFCC

Note: elementary functions are ‘usually’ continuous. Find an elementary function which is discontinuous at some point.

Hint: keep a close watch on Th. 7.

Theorem 9. If function f is continuous at a and $f(a) > 0$, then there exists a nbd of a such that for each x from this nbd we have $f(x) > 0$, i.e.:

$$\exists_{\delta>0}\forall_{x\in D}\, |x-a|<\delta \Rightarrow f(x)>0.$$

Proof:

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x\in D}\left[|x-a|<\delta \Rightarrow |f(x)-f(a)|<\varepsilon\right]$$

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x\in D}\left[|x-a|<\delta \Rightarrow \underbrace{f(a)-\varepsilon}_{\text{positive}} < f(x) < \underbrace{f(a)+\varepsilon}_{\text{positive}}\right]$$

Let $\varepsilon < f(a)$

■

Theorem 10 (Intermediate Value Theorem). If function f is continuous on $\langle a, b\rangle$, then f has the Darboux property on $\langle a, b\rangle$. That is, if d is any value between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = d$.

Note: discontinuous functions also may have the Darboux property.

Theorem 11. If function f is continuous on $\langle a, b\rangle$ and $f(a)f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Proof: Directly from Th.10. ■

Note: explain the practical meaning of this theorem.

Theorem 12 (the Weierstrass Theorem). If function f is continuous on $\langle a, b\rangle$, then it takes on both global minimum and global maximum values in the interval.

⇓ Th.10

Corollary 1. If function f is continuous on $\langle a, b\rangle$ then it bounded.

Corollary 2. If function f is continuous on $\langle a, b\rangle$ then its range is equal to $\langle m, M\rangle$ where m and M are global minimum and global maximum, respectively.

Find an example of a function f such that $R_f \neq \langle m, M\rangle$.

Df. 1. The oscillation of function f is defined as the difference between its global maximum and global minimum:

$$\omega = M - m$$

Df. 2. The partition of $\langle a, b\rangle$ is defined as the set

$$P_n = \{\langle x_0, x_1\rangle, \langle x_1, x_2\rangle, \langle x_2, x_3\rangle, \dots, \langle x_{n-1}, x_n\rangle\}$$

where $x_0, x_1, x_2, \dots, x_n$ are arbitrarily chosen points such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Theorem 13. If function f is continuous on $\langle a, b\rangle$, then there exists the partition of $\langle a, b\rangle$ such that the oscillation in each subinterval is less than any arbitrarily chosen positive number, i.e.:

$$\forall_{\varepsilon>0} \exists_{n\in\mathbb{N}} \forall_{1\le i\le n} \ \omega_i = M_i - m_i < \varepsilon.$$

Asymptotes

Df. 3. The curve $y = f(x)$ has vertical asymptote $x = a$ iff

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty.$$

Df. 4. The curve $y = f(x)$ has horizontal asymptote $y = b$ iff

$$\lim_{x \to +\infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

Df. 5. The curve $y = f(x)$ has slant asymptote $y = ax + b$ iff

$$\lim_{x\to+\infty}\left[f(x)-(ax+b)\right]=0 \quad \text{or} \quad \lim_{x\to-\infty}\left[f(x)-(ax+b)\right]=0.$$

Theorem 14. The line $y = ax + b$ is the slant asymptote of curve $y = f(x)$ iff

$$\begin{cases} a = \lim\limits_{x\to+\infty}\dfrac{f(x)}{x} \\ b = \lim\limits_{x\to+\infty}[f(x)-ax] \end{cases} \quad \text{or} \quad \begin{cases} a = \lim\limits_{x\to-\infty}\dfrac{f(x)}{x} \\ b = \lim\limits_{x\to-\infty}[f(x)-ax]. \end{cases}$$

IS IT POSSIBLE THAT THE GRAPH OF A FUNCTION

- has infinitely many vertical asymptotes?
- has 3 different horizontal asymptotes?
- has simultaneously vertical, horizontal and slant asymptotes?
- intersects its asymptote?
- has vertical asymptote if $D_f = \mathbb{R}$?