PART 6

DERIVATIVES AND DIFFERENTIALS

Definition of derivative

Now we consider real-valued functions of one real variable.

Df. 1. The derivative of function f at a point a is defined as

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

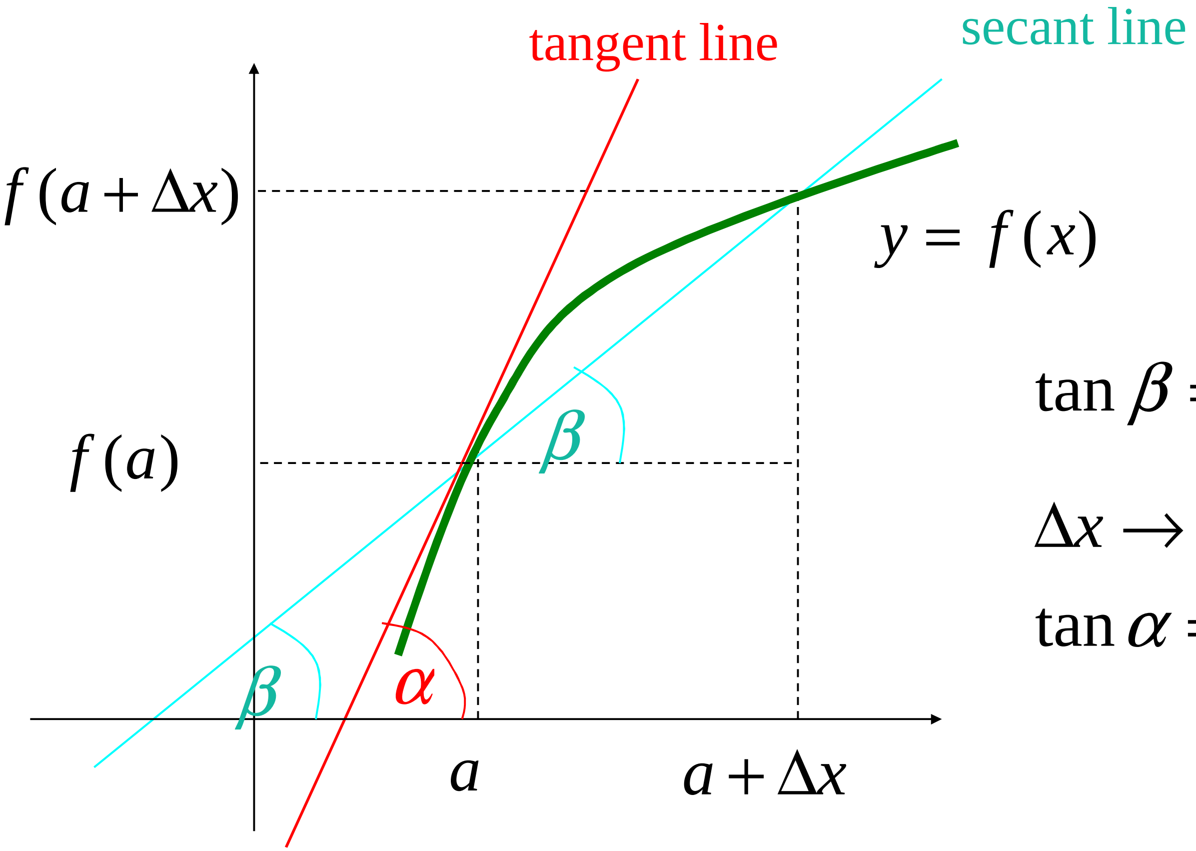
if the limit exists and is finite.

If $f'(a)$ exists, then we say that f is differentiable at a.

Notation:

1) if $y = f(x)$ is differentiable at a, then $f'(a) = \left.\frac{dy}{dx}\right|_a = \left.\frac{df}{dx}\right|_a$

2) $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$



$$\tan\beta = \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

$$\Delta x \to 0 \Rightarrow \beta \to \alpha$$

$$\tan\alpha = f'(a)$$

Tangent line – the limit position of secant lines.

Normal line – the line perpendicular to the tangent line at the point of tangency.

Tangent line:

$$y - f(a) = f'(a)(x - a)$$

Normal line:

$$y - f(a) = -\frac{1}{f'(a)}(x - a) \quad \text{if} \quad f'(a) \neq 0$$

$$x = a \quad \text{if} \quad f'(a) = 0$$

Suppose that $f'(a)$ exists. Then

$$\lim_{x\to a} f(x) = \begin{bmatrix} x-a=\Delta x \\ x = a+\Delta x \\ x\to a \Rightarrow \Delta x \to 0 \end{bmatrix} = \lim_{\Delta x\to 0} f(a+\Delta x) =$$

$$= \lim_{\Delta x\to 0}\left[f(a+\Delta x) - f(a) + f(a)\right] =$$

$$= \lim_{\Delta x\to 0}\left[\underbrace{\frac{f(a+\Delta x)-f(a)}{\Delta x}}_{f'(a)}\overbrace{\Delta x}^{0} + \underbrace{f(a)}_{f(a)}\right] = f(a)$$

WHAT DOES IT MEAN?

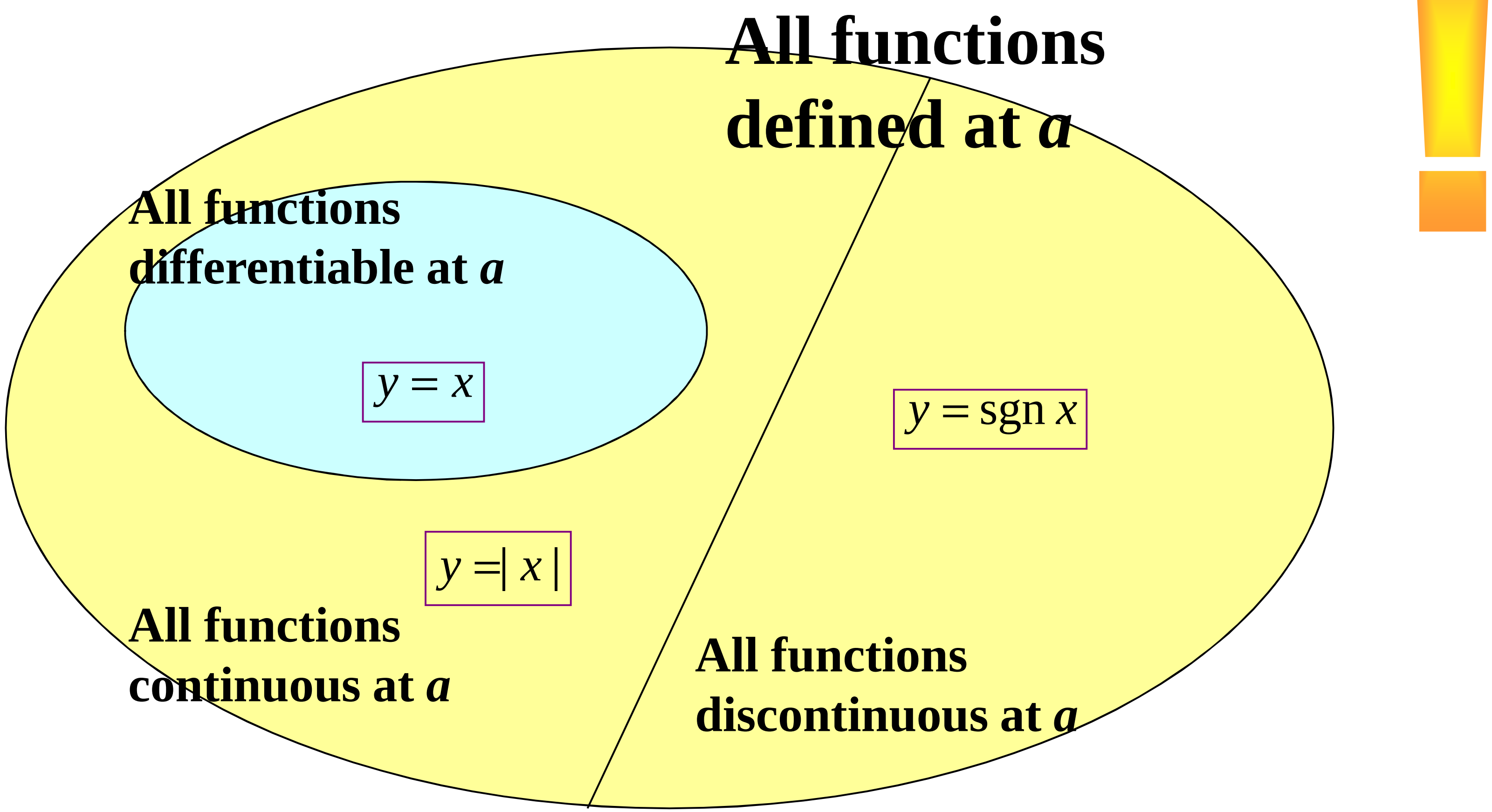
Example: find the derivative of $f(x) = |x|$ at 0.

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\Delta x) - |0|}{\Delta x} = \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x} = ?$$

$$\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = \lim_{\Delta x \to 0^+} 1 = 1;$$

$$\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{-\Delta x}{\Delta x} = \lim_{\Delta x \to 0^+} (-1) = -1.$$

Hence $f'(0)$ does not exist.



Note: differentiable functions are also called smooth functions.

Continuity is necessary but not sufficient for differentiability.

Differentiability is sufficient but not necessary for continuity.

Df. 2. If for all $x \in D' \subset D$, the numer $f'(x)$ exists, then we can build the new function

$$f' : y = f'(x), \qquad x \in D'$$

called the derivative of f.

Note: $f'(x)$ – a number

f' – a function

Properties of differentiable functions

Theorem 1. If f and g are differentiable functions, and c is a real constant, then:

1) $(cf)' = cf'$

2) $(f+g)' = f' + g'$

3) $(fg)' = f'g + fg'$

4) $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$ (if $g \neq 0$).

Proof → AFCC

Theorem 2. If $y = f(x)$ is strictly monotonic in an interval I, and there exists a point $a \in I$ such that $f'(a) \neq 0$, then the derivative of inverse function $x = f^{-1}(y)$ exists at the point $f(a)$ and is equal to $\frac{1}{f'(a)}$.

In other words, $\left.\frac{dx}{dy}\right|_{y_0} = \frac{1}{\left.\frac{dy}{dx}\right|_{x_0}}$ or, equvalently, $\left.\frac{dy}{dx}\right|_{x_0} = \frac{1}{\left.\frac{dx}{dy}\right|_{y_0}}$

$(x_0 = a, y_0 = f(a))$.

Theorem 3. (Chain Rule) If $u = g(x)$ is differentiable at x_0 and $y = f(u)$ is differentiable at $u_0 = f(x_0)$, then the composite function $y = f(g(x))$ is differentiable at x_0 and

$$(f \circ g)'(x_0) = f'(u_0)g'(x_0).$$

In other words, $\left.\frac{dy}{dx}\right|_{x_0} = \left.\frac{dy}{du}\right|_{u_0} \cdot \left.\frac{du}{dx}\right|_{x_0}$.

Table of derivatives

Derive each formula!

$$(c)' = 0$$

$$(x)' = 1$$

$$(ax+b)' = a$$

$$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$$(x^s)' = sx^{s-1}, s \in \mathbb{R}$$

$$(e^x)' = e^x$$

$$(a^x)' = a^x \ln a \quad (a > 0)$$

$$(\ln x)' = \frac{1}{x}$$

$$(\log_a x)' = \frac{1}{x \ln a} \quad (a > 0, a \neq 1)$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \frac{1}{\cos^2 x}$$

$$(\cot x)' = -\frac{1}{\sin^2 x}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

$$(\sinh x)' = \cosh x$$

$$(\cosh x)' = \sinh x$$

$$(\tanh x)' = \frac{1}{\cosh^2 x}$$

$$(\coth x)' = -\frac{1}{\sinh^2 x}$$

Differentials

Df. 3. If f is differentiable and $\Delta x \neq 0$ is an increment of independent variable, then the expression $dy = f'(x)\Delta x$ is called the differential of *f*.

Remark 1:

$y = x$

$dy = 1 \cdot \Delta x = \Delta x$

$dx = \Delta x$

Remark 2:

$$dy : dx = \frac{dy}{dx} = \frac{y'\Delta x}{\Delta x} = y'$$

Theorem 4. If f is differentiable at x, then

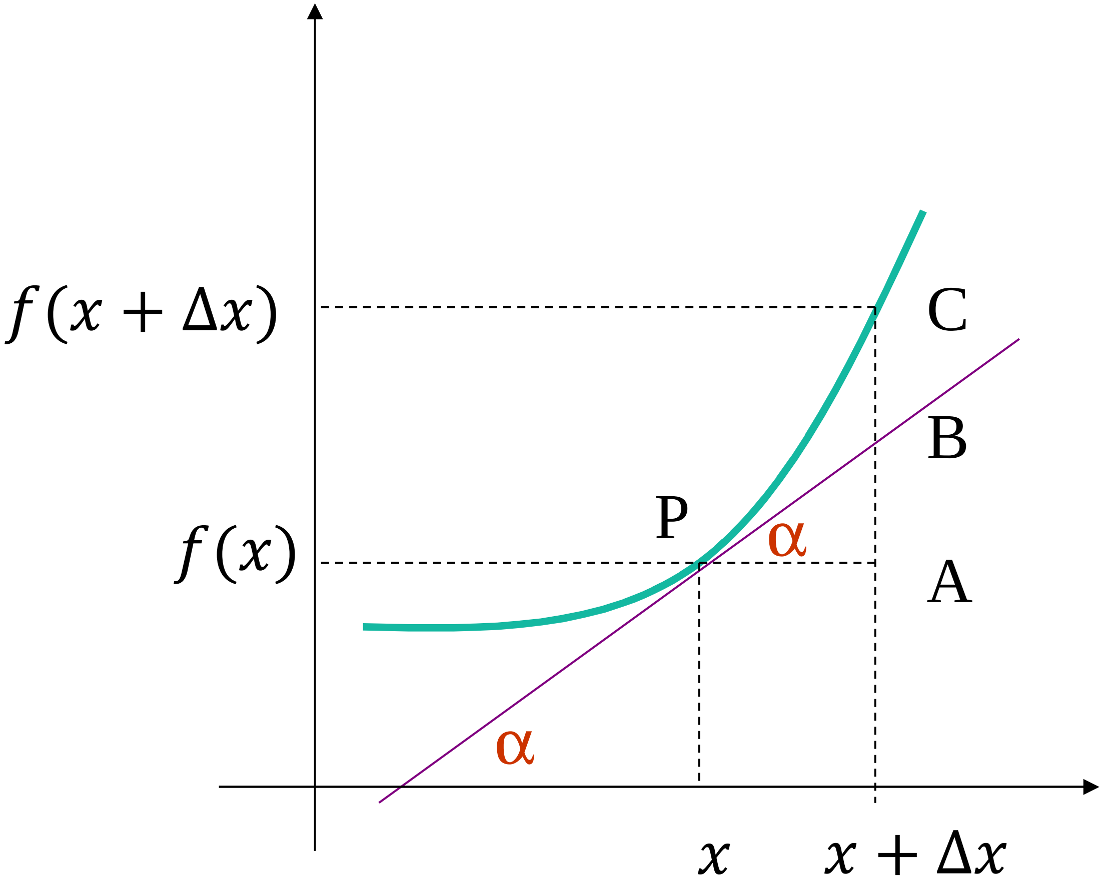
$$\Delta y = f(x+\Delta x) - f(x) = dy + \varepsilon\Delta x, \text{ where } \lim_{\Delta x\to 0} \varepsilon = 0.$$

Proof:

$$\underbrace{\lim_{\Delta x\to 0}\frac{\Delta y - dy}{\Delta x}}_{=\varepsilon} = \lim_{\Delta x\to 0}\frac{\Delta y - y'\Delta x}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x} - y'\right) = 0$$

($\frac{\Delta y}{\Delta x} \to y'$) ∎

Geometrical interpretation of differential



$$AP = \Delta x$$

$$AC = \Delta y$$

$$f'(x) = \tan\alpha$$

$$\tan\alpha = \frac{AB}{AP}$$

$$AB = f'(x)\Delta x = dy$$

$$BC = \Delta y - dy = \varepsilon\Delta x$$

Application of Th. 4:

$$\text{if } \Delta x \approx 0, \text{ then } \Delta y \approx dy, \text{ i.e.}$$

$$f(x+\Delta x) \approx f(x) + dy.$$

approximate calculations

Properties of differentials

Theorem 5. If u, v are differentiable, and c is a real constant, then

1) $d(cu) = cdu$

2) $d(u+v) = du + dv$

3) $d(uv) = vdu + udv$

4) $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$ (if $v \neq 0$)

Proof → AFCC

Derivative of function given by parametric equations

$$\begin{cases} x = x(t) \\ y = y(t), t \in I \end{cases}$$

$$t - \text{time} \Rightarrow \{(x(t), y(t)) : t \in I\} - \text{a curve}$$

If x and y are differentiable, x' is different than 0, then the set above is the graph of some function $y = f(x)$. Moreover,

$$f'(x) = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)} \overset{notation}{=} \frac{\dot{x}}{\dot{y}}$$

Derivatives of higher orders

Df. 4. $$f^{(n)} \overset{\text{df}}{=} \left(f^{(n-1)}\right)' \overset{\text{notation}}{=} \frac{d^n f}{dx^n}, \quad n \geq 2.$$

Example:

$$f(x) = e^x, \ f'(x) = e^x, \ f''(x) = e^x, \ f'''(x) = e^x, \ f^{(4)}(x) = e^x, \dots,$$
$$f^{(n)}(x) = e^x, \dots$$

Example:

$$f(x) = \sin x, \ f'(x) = \cos x, \ f''(x) = -\sin x, \ f'''(x) = -\cos x,$$
$$f^{(4)}(x) = \sin x, \dots$$

Hence $$f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right), \quad n \in \mathbf{N}.$$

Differentials of higher orders

$$\Delta x = dx \neq 0 \text{ constant}$$

$$dy = y' dx - \text{a function of } x$$

$$d^2 y \overset{\text{df}}{=} d(dy) = d(y' dx) = (y' dx)' dx =$$

$$= y'' dxdx = y''(dx)^2 \overset{\text{notation}}{=} y'' dx^2$$

$$d^3 y \overset{\text{df}}{=} d(d^2 y) = d(y'' dx^2) = (y'' dx^2)' dx =$$

$$= y''' dx^2 dx = y'''(dx)^3 \overset{\text{notation}}{=} y''' dx^3, \text{ etc.}$$

$$d^n y \overset{\text{df}}{=} d(d^{n-1}y) = \ldots = y^{(n)} dx^n \quad \Rightarrow \quad y^{(n)} = \frac{d^n y}{dx^n}$$

Remark:

$$dx^n = (dx)^n = dx \cdot dx \cdot \ldots. \cdot dx$$

$$d(x^n) = nx^{n-1} dx$$

$$d^n x = d(d^{n-1}x) = \begin{cases} dx & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \geq 2 \end{cases}$$