PART 7

APPLICATIONS OF DERIVATIVES (1)

Fermat's Theorem

Theorem 1. Let f be a function defined on (a, b) which attains its global maximum or minimum at $c \in (a, b)$. If f is differentiable at c, then $f'(c) = 0$.

Proof:

1° $[f(c) = M \Leftrightarrow \forall_x \; f(x) \le M] \Rightarrow \forall_x \; f(x) - f(c) \le 0$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists}$$

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0$$

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0$$

$$f'(c) = 0.$$

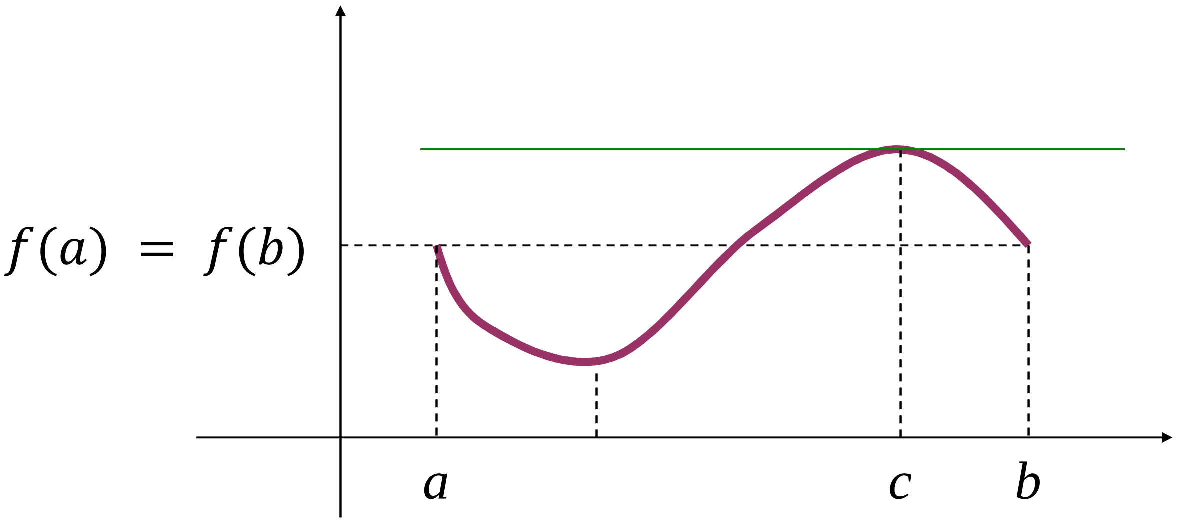
2° $f(c) = m$

analogously

Rolle's Theorem

Theorem 2. If a function f is continuous on $\langle a, b\rangle$, differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof: by Fermat's Theorem.... ∎



Lagrange's Theorem

Theorem 3. If a function f is continuous on $\langle a, b\rangle$, differentiable on (a, b), then there exists $c \in (a, b)$ such that

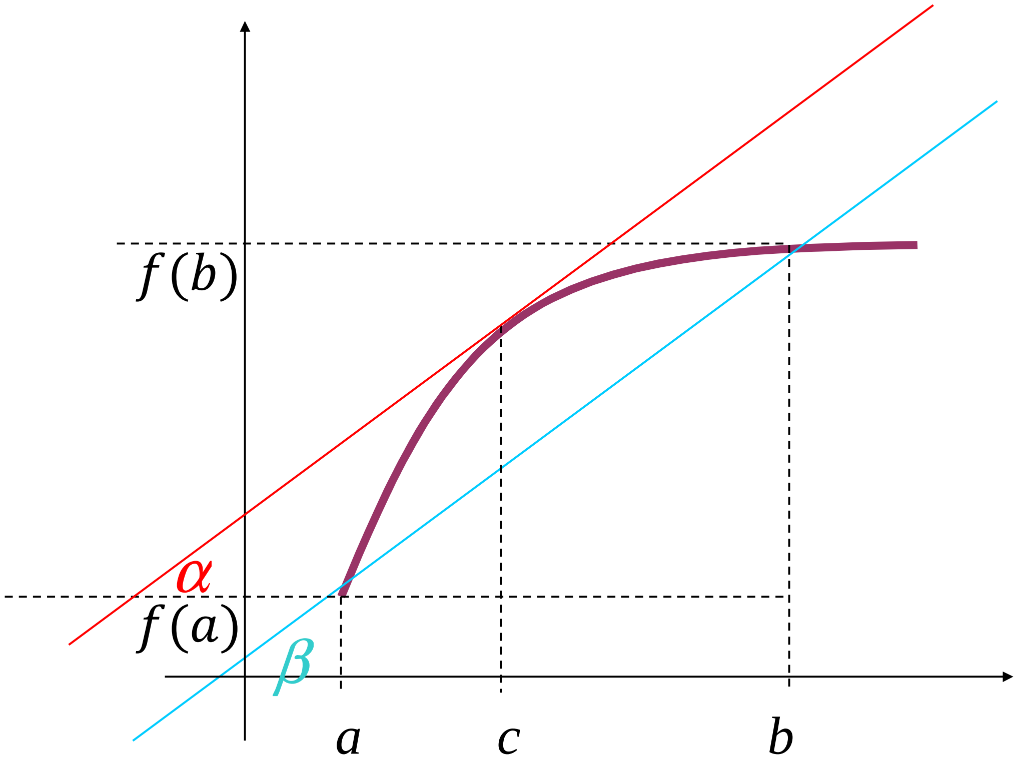
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof:

Apply the Rolle Theorem to

$$\varphi(x) = \frac{f(b) - f(a)}{b - a}(x - a) - f(x) + f(a)$$

■



$$\tan\beta = \frac{f(b) - f(a)}{b - a}$$

$$\tan\alpha = f'(c)$$

$$\tan\alpha = \tan\beta$$

Corollaries from the Lagrange Theorem

1) $\left[\forall_{x\in(a,b)}\ f'(x)=0\right] \Rightarrow f(x)=\text{const}$

2) $\left[\forall_{x\in(a,b)}\ f'(x)=g'(x)\right] \Rightarrow \exists_A \forall_x\ f(x)=g(x)+A$

3) $\left[\forall_{x\in(a,b)}\ f'(x)>0\right] \Rightarrow f$ is increasing on (a,b)

4) $\left[\forall_{x\in(a,b)}\ f'(x)<0\right] \Rightarrow f$ is decreasing on (a,b)

WHY?

WHAT ABOUT CONVERSE IMPICATIONS?

Extreme values

Df. 1. Function f has local minimum at a iff

$$\exists_{\delta>0}\forall_{x\in(a-\delta,a)\cup(a,a+\delta)}\ f(x)>f(a).$$

Value $f(a)$ is called then the local minimum of f.

Df. 2. Function f has local maximum at a iff

$$\exists_{\delta>0}\forall_{x\in(a-\delta,a)\cup(a,a+\delta)}\ f(x)<f(a).$$

Value $f(a)$ is called then the local maximum of f.

Necessary conditions for extreme values

$f(a)$ is the local extreme value of f

f is differentiable at a

by Fermat's Theorem

$$f'(a) = 0$$

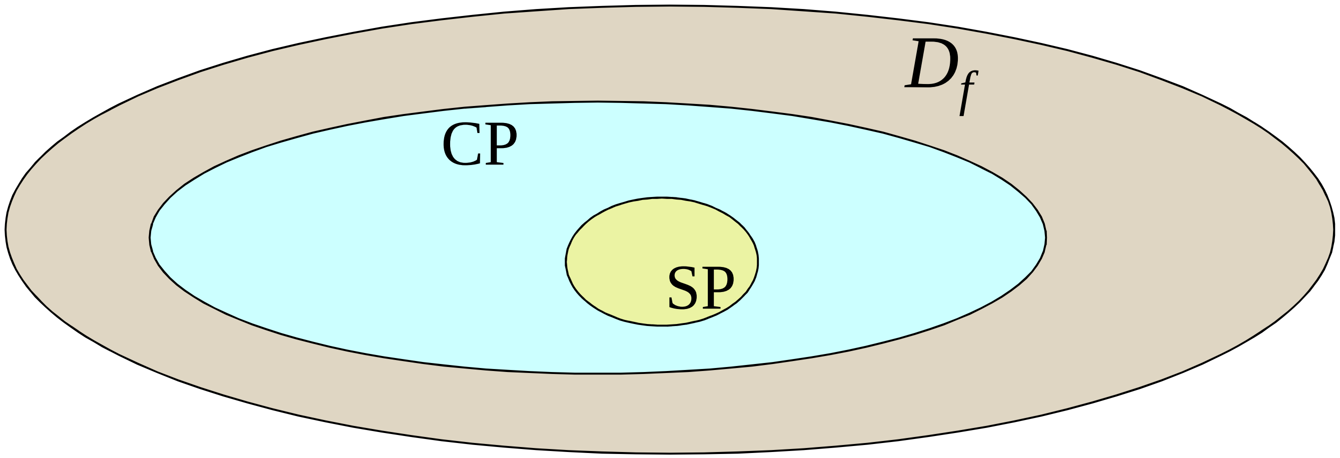
f is not differentiable at a

$f'(a)$ does not exist

$$a \in D_f$$

$f'(a) = 0 \Rightarrow$ a is called the stationary point of f

$[\, f'(a) = 0 \vee f'(a)$ doesn't exist$] \Rightarrow$ a is called the critical point of f



Example:

$$f(x) = -(x-1)^2$$

f has local maximum 0 at 1

$$f'(1) = 0$$

$$g(x) = 2 + |x|$$

g has local minimum 2 at 0

$$g'(0) - \text{does not exist}$$

Theorem 6. If function f has local extreme value at a, then a is a critical point of f.

Note: necessary conditions are not sufficient.

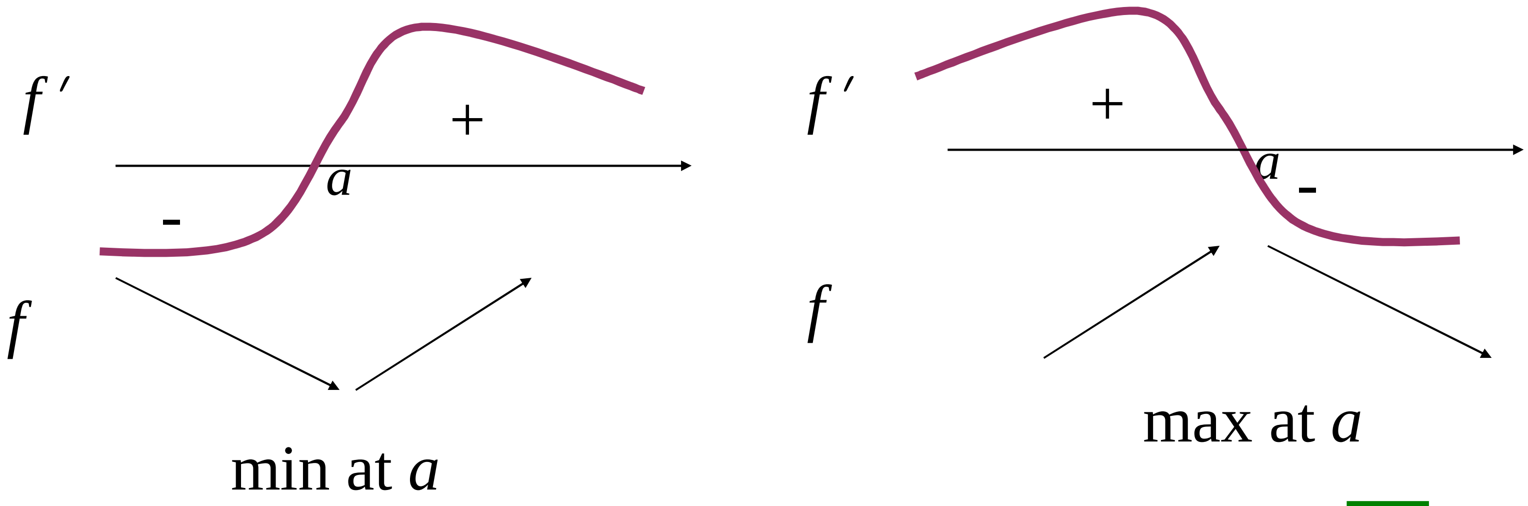
Example:

$f(x) = x^3$ $f'(x) = 3x^2$ $f'(x) = 0 \Leftrightarrow x = 0$ f doesn't have local extreme value at 0	$g(x) = \operatorname{sgn} x$ $g'(0)$ – doesn't exist g doesn't have local extreme value at 0

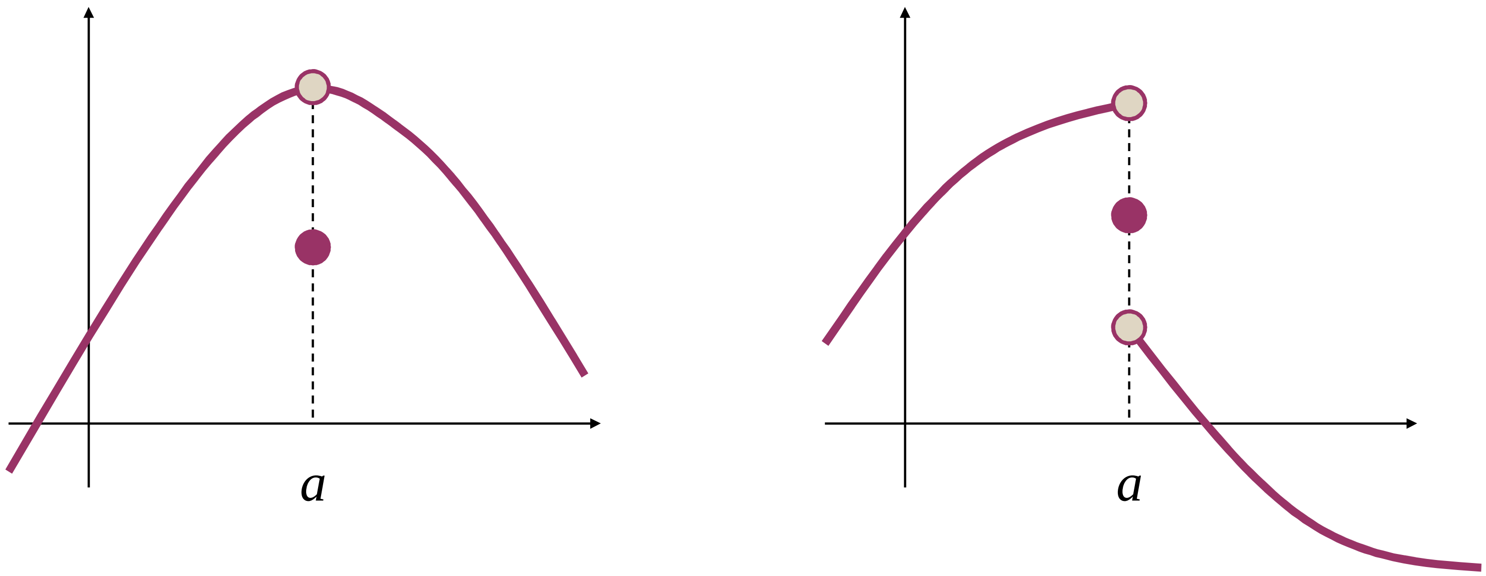
Sufficient conditions for extreme value

Theorem 7. If function f is continuous at a critical point a and f' changes its sign passing through a, then f has local extreme value at a.

Proof



Note: the continuity at a in Th. 7 is important.



Cauchy's Theorem

Theorem 4. If a functions f and g are continuous on $\langle a, b\rangle$, differentiable on (a, b), and $g'(x) \neq 0$ for each $x \in (a, b)$, then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof:

Apply the Rolle Theorem to

$$\varphi(x) = \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)] - f(x) + f(a)$$ ■

Taylor's Formula

Theorem 5. If $f \in C^{n-1}$ on the interval $\langle x_0, x \rangle$ and $f^{(n)}$ exists on (x_0, x), then there exists c strictly between x_0 and x such that

$$f(x) =$$

$$= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3$$

$$+ \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + \frac{f^{(n)}(c)}{n!}(x - x_0)^n.$$

Proof AFCC

Note: The same theorem is true if we replace intervals $\langle x_0, x \rangle$ and (x_0, x) with $\langle x, x_0 \rangle$ and (x, x_0), respectively.

If $f^{(n)}$ is bounded on (x_0, x) or (x, x_0) then:

$$|R_n| = \left| \frac{f^{(n)}(c)}{n!}(x-x_0)^n \right| \le M \frac{|x-x_0|^n}{n!} \xrightarrow{n\to\infty} 0$$

so $f(x) \approx T_{n-1}$ for n sufficiently large

Approximation by the Taylor polynomial is the best!

$x_0 = 0$ ⟹ Maclaurin's Formula

$$c = x_0 + \theta(x - x_0),\ \text{where } 0 < \theta < 1$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(\theta x)}{n!}x^n$$

$$f(x) = e^x \qquad f(0) = 1$$

$$f^{(k)}(x) = e^x \qquad f^{(k)}(0) = 1$$

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots + \frac{1}{(n-1)!}x^{n-1} + \frac{e^{\theta x}}{n!}x^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{e^{\theta x}}{n!}x^n, \; x \in \mathbf{R}$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(\theta x)}{n!}x^n$$

$$f(x) = \cos x$$

$$f'(x) = -\sin x \qquad f^{(k)}(x) = \cos\left(x + k\frac{\pi}{2}\right)$$

$$f''(x) = -\cos x$$

$$f'''(x) = \sin x$$

$$f(0) = 1 \qquad f^{(2k)}(0) = (-1)^k$$

$$f'(0) = 0$$

$$f''(0) = -1 \qquad f^{(2k+1)}(0) = 0$$

$$f'''(0) = 0$$

$$\cos x = 1 + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \ldots + \frac{(-1)^n}{(2n)!}x^{2n} +$$

$$+ \frac{0}{(2n+1)!}x^{2n+1} + \frac{\cos\left(x + (2n+2)\frac{\pi}{2}\right)}{(2n+2)!}x^{2n+2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^n}{(2n)!}x^{2n} + \frac{\cos(\theta x + (n+1)\pi)}{(2n+2)!}x^{2n+2}, \quad x \in \mathbf{R}$$

Show (analogously) that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1} + \\ + \frac{\sin\left(\theta x + (2n+1)\frac{\pi}{2}\right)}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbf{R}$$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(\theta x)}{n!}x^n$$

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f'''(x) = \frac{1\cdot 2}{(1+x)^3}$$

$$f^{(4)}(x) = -\frac{1\cdot 2\cdot 3}{(1+x)^4}$$

$$f^{(k)}(x) = (-1)^{k+1}\frac{(k-1)!}{(1+x)^k}$$

$$f(0) = \ln 1 = 0$$

$$f^{(k)}(0) = (-1)^{k+1}(k-1)!$$

$$\ln(1+x) = 0 + \frac{0!}{1!}x - \frac{1!}{2!}x^2 + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 \ldots + (-1)^n \frac{(n-2)!}{(n-1)!}x^{n-1} +$$

$$+ (-1)^{n+1}\frac{(n-1)!}{(1+\theta x)^n n!}x^n$$

$$\ln(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots + \frac{(-1)^n x^{n-1}}{n-1} + \frac{(-1)^{n+1}x^n}{n(1+\theta x)^n},\ x > -1$$

$$f(x) = (1+x)^s,\ s \in \mathbf{R}$$

$$(1+x)^s = 1 + \frac{s}{1!}x + \frac{s(s-1)}{2!}x^2 + \frac{s(s-1)(s-2)}{3!}x^3 +$$

$$+ \ldots + \frac{s(s-1)(s-2)\ldots(s-n+2)}{(n-1)!}x^{n-1} +$$

$$+ \frac{s(s-1)(s-2)\ldots(s-n+2)(s-n+1)}{n!}(1+\theta x)^{s-n}x^n,$$

$$x > 1$$

Theorem 8. Let $n, k \in \mathbb{N}$ and $f \in C^n(\text{nbd } a)$. Then:

1) $[n = 2k \wedge f'(a) = f''(a) = \ldots = f^{(2k-1)}(a) = 0 \wedge$

$\wedge \ f^{(2k)}(a) > 0] \Rightarrow f(a) = f_{\min}$

2) $[n = 2k \wedge f'(a) = f''(a) = \ldots = f^{(2k-1)}(a) = 0 \wedge$

$\wedge \ f^{(2k)}(a) < 0] \Rightarrow f(a) = f_{\max}$

3) $[n = 2k - 1 \wedge f'(a) = f''(a) = \ldots = f^{(2k-2)}(a) = 0 \wedge$

$\wedge \ f^{(2k-1)}(a) \neq 0] \Rightarrow f$ does not have extreme value at a.

Proof: it follows from the Taylor Formula. ∎

Example:

$$f(x)=\frac{x^5}{5}-\frac{x^3}{3}+\frac{1}{3}, \quad D=\mathbb{R}$$

$$f'(x)=x^4-x^2, \qquad D'=\mathbb{R}$$

$$f'(x)=0 \Leftrightarrow x^4-x^2=0 \Leftrightarrow x^2(x^2-1)=0 \Leftrightarrow x\in\{1,-1,0\}$$

$$f''(x)=4x^3-2x, \quad D''=\mathbb{R}$$

$$f''(1)=4-2>0 \Rightarrow \boxed{f_{\min}=f(1)=\frac{1}{5}}$$

$$f''(-1)=-4+2<0 \Rightarrow \boxed{f_{\max}=f(-1)=\frac{4}{15}}$$

$$f''(0)=0;\ f'''(x)=12x-2$$

$$f'''(0)\neq 0 \Rightarrow \boxed{f \text{ does not have extreme value at } 0.}$$