# PART 8

# APPLICATIONS OF DERIVATIVES (2)

# Concavity

**Df. 3.** A function $f$ continuous on interval $I$ (open or closed, bounded or unbounded) is called concave up (or concave upward) on $I$ iff

$$\forall_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \forall_{\lambda \in (0,1)} \; f\left(\lambda x_1 + (1-\lambda) x_2\right) < \lambda f(x_1) + (1-\lambda) f(x_2).$$

**Df. 4.** A function $f$ continuous on interval $I$ (open or closed, bounded or unbounded) is called concave down (or concave downward) on $I$ iff

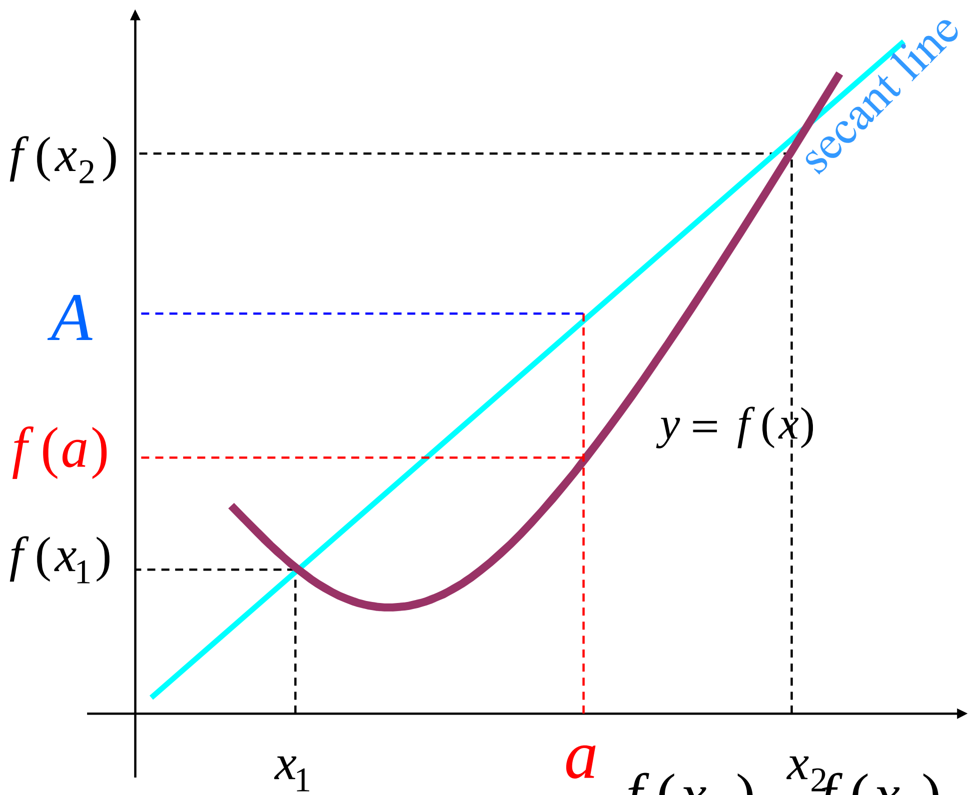
$$\forall_{\substack{x_1, x_2 \in I \\ x_1 \neq x_2}} \forall_{\lambda \in (0,1)} \; f\left(\lambda x_1 + (1-\lambda) x_2\right) > \lambda f(x_1) + (1-\lambda) f(x_2).$$

Note: if $f$ is CU, then $g = -f$ is CD.

$$0 < \lambda < 1 \Rightarrow \lambda x_1 + (1 - \lambda x_2) = a \in (x_1, x_2)$$

$$f(a) = LHS$$

$$\text{secant line}: \; y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$




$$LHS < RHS$$

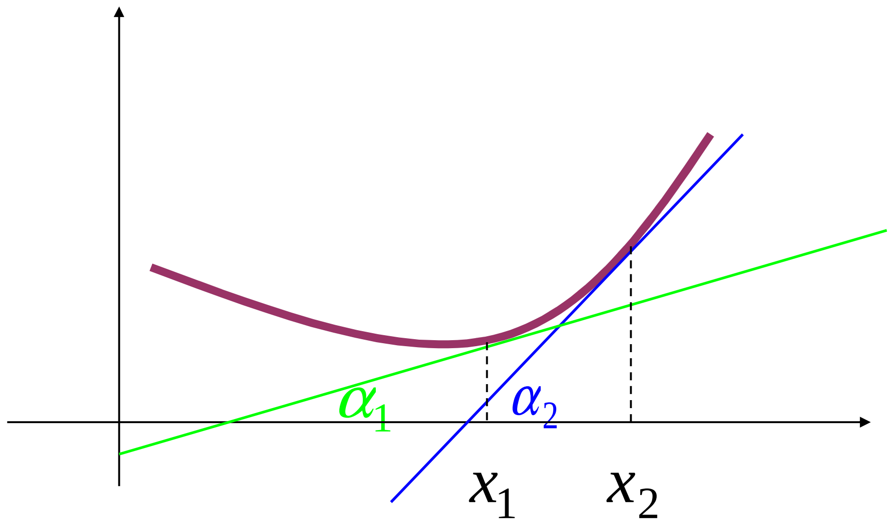
⇓

$f$ is CU

$$x = a \Rightarrow A = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(a - x_1) = \ldots$$

$$= \lambda f(x_1) + (1 - \lambda) f(x_2) \Rightarrow \quad A = RHS$$

Suppose that $f$ is CU and twice differentiable on interval $I$.




$$x_1 < x_2$$
$$\Downarrow$$
$$\alpha_1 < \alpha_2$$
$$\tan\alpha_1 < \tan\alpha_2$$
$$f'(x_1) < f'(x_2)$$

$$[f \text{ is CU and } f'' \text{ exists on } I] \Rightarrow f' \text{ is incr. on } I \Rightarrow$$
$$\Rightarrow \forall_{x\in I}\; f''(x) \geq 0.$$

$$[\forall_{x\in I}\; f''(x) > 0] \Rightarrow f \text{ is CU on } I.$$

**WHY?**

**Df. 5.** If function $f$ is CU (or CD) on interval $I$, then its graph is called concave up (or concave down) .

**Df. 6**. If $f$ is continuous at $a$, and $f$ is

- CU on $(a - \delta, a)$ and CD on $(a, a + \delta)$

or

- CD on $(a - \delta, a)$ and CU on $(a, a + \delta)$, where $\delta > 0$

then $a$ is called inflection number of $f$.

**Df. 7.** If $a$ is the inflection number of function $f$, then the point $P(a, f(a))$ is called the inflection point of curve $y = f(x)$.

**Theorem 9.** If $a$ is the inflection number of $f$, then $f''(a) = 0$ or $f''(a)$ does not exist.

**Theorem 10.** If $f''(a) = 0$ and the second derivative changes its sign passing through $a$, then $a$ is the inflection number of $f$.

# De l'Hospital's Rules

**Theorem 11 (de l'Hospital's Rule).** Let functions *f, g* satisfy the following conditions:

1) $f, g$ are continuous on $(a, b\rangle$

2) $f, g$ are differentiable on $(a, b)$

3) $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x$ from some nbd of $a$

4) $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$

5) $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists (proper or improper).

Then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$

## Proof:

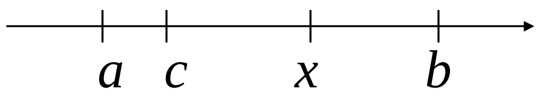
Let us consider functions $F$ and $G$ defined as:

$$F(x)=\begin{cases} f(x) & \text{if} \quad x\in(a,b\rangle \\ 0 & \text{if} \quad x=a \end{cases} \qquad G(x)=\begin{cases} g(x) & \text{if} \quad x\in(a,b\rangle \\ 0 & \text{if} \quad x=a \end{cases}$$

We can apply the Cauchy Theorem to $F$ and $G$ on interval $\langle a,x\rangle$, where $x\in(a,b)$:

**verify assumptions!**

$$\frac{f(x)}{g(x)}=\frac{f(x)-0}{g(x)-0}=\frac{F(x)-F(a)}{G(x)-G(a)}=\frac{F'(c)}{G'(c)}=\frac{f'(c)}{g'(c)}$$




$$\lim_{x\to a^+}\frac{f(x)}{g(x)}=\lim_{\substack{(x\to a^+)\\ c\to a^+}}\frac{f'(c)}{g'(c)}=\lim_{x\to a^+}\frac{f'(x)}{g'(x)}.$$

# Remarks:

- we can state and prove analogously theorem for limits from the left or at infinity;

- Th. 11 holds for indeterminate form $\left[\frac{0}{0}\right]$ as well as $\left[\frac{\infty}{\infty}\right]$ (the proof for last case is not so easy);

- to apply de l'Hospital's Rules we have to be sure that the limit of quotient of derivatives exists.

# Indeterminate forms

$$\frac{f}{g}\left[\frac{0}{0},\frac{\infty}{\infty}\right]$$ → directly de l'Hospital's Rules

$$fg\ [0\cdot\infty] = \frac{f}{1/g}\left[\frac{0}{0}\right] = \frac{g}{1/f}\left[\frac{\infty}{\infty}\right]$$

$$f-g\ [\infty-\infty] = \frac{1}{1/f} - \frac{1}{1/g} = \frac{1/g-1/f}{1/(fg)}\left[\frac{0}{0}\right]$$

$$f^{g}\ [1^{\infty},0^{0},\infty^{0},] = e^{\ln f^{g}} = e^{g\ln f}$$

$g\ln f$ → $[0\cdot\infty]$

# Curve sketching

1. Domain; *x*-intercept, *y*-intercepts; is the function even, odd, periodic?
2. Limits at the endpoints of domain; asymptotes.
3. Analysis of the first derivative (monotonicity, extreme values).
4. Analysis of the second derivative (concavity, inflections).
5. Collection of all informations in the table.
6. Sketching the curve.