PART 9

INDEFINITE INTEGRALS (1)

Antiderivative

Df. 1. A function F is called an antiderivative of function f on interval I iff

$$\forall_{x\in I} \; F'(x) = f(x).$$

Example:

$$f(x) = \cos x + 2x \quad \Longrightarrow \quad F(x) = \sin x + x^2$$

$$\text{or } F(x) = \sin x + x^2 + 1$$

$$\text{or } F(x) = \sin x + x^2 - \ln 3, \text{ etc.}$$

Theorem 1. If F is an antiderivative of f, then each function of the form $G(x) = F(x) + C$ (where C is any real constant) is also an antiderivative of f.

Proof:

$$F'(x) = f(x)$$

$$G'(x) = [F(x) + C]' = F'(x) + 0 = f(x).$$

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Theorem 2. If F and G are antiderivatives of f, then there exists a real constant C such that $G(x) = F(x) + C$.

Proof:

$$F'(x) = f(x) = G'(x)$$

$\Downarrow$ by 2nd corollary from the Lagrange Theorem

$$\exists_{C\in\mathbf{R}}\ G(x) = F(x) + C. \qquad \blacksquare$$

By Th. 1 and 2,

if we know one antiderivative of f,
then we know all antiderivatives of f.

Indefinite integral

Df. 2. The indefinite integral of function f is defined as

$$\int f(x)dx = F(x) + C$$

where $F'(x) = f(x)$ and $C \in \mathbf{R}$ is arbitrary constant.

$$\int f(x)dx$$

integral sign

integrand

variable of integration

Theorem 3. Any continuous function possesses antiderivative (not necessarily given by elementary function). A discontinuous function may possess antiderivative.

Note: this theorem will be proven later (using definite integrals)

Table of basic integrals

1) $\int 0dx = C$

2) $\int 1dx = \int dx = x + C$

3) $\int x^s dx = \frac{x^{s+1}}{s+1} + C \quad (s \neq -1)$

4) $\int \frac{1}{x} dx = \int \frac{dx}{x} = \ln|x| + C$

5) $\int \frac{1}{1+x^2} dx = \int \frac{dx}{1+x^2} = \arctan x + C$

6) $\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

7) $\int e^x dx = e^x + C$

8) $\int a^x dx = \frac{a^x}{\ln a} + C \ (a > 0,\ a \neq 1)$

9) $\int \sin x dx = -\cos x + C$

10) $\int \cos x dx = \sin x + C$

11) $\int \frac{1}{\cos^2 x} dx = \int \frac{dx}{\cos^2 x} = \tan x + C$

12) $\int \frac{1}{\sin^2 x} dx = \int \frac{dx}{\sin^2 x} = -\cot x + C$

13) $\int \sinh x dx = \cosh x + C$

14) $\int \cosh x dx = \sinh x + C$

15) $\int \frac{1}{\cosh^2 x} dx = \int \frac{dx}{\cosh^2 x} = \tanh x + C$

16) $\int \frac{1}{\sinh^2 x} dx = \int \frac{dx}{\sinh^2 x} = -\coth x + C$

Keep *C* and *dx* in mind.

$$\int 2xy\,dx = x^2 y + C, \ \int 2xy\,dy = xy^2 + C, \ \int 2xy\,dt = 2xyt + C$$

Properties of indefinite integrals

Theorem 4. If f is continuous, then $d\left(\int f(x)dx\right) = f(x)dx.$

Proof:

$$\int f(x)dx = F(x) + C \quad \text{where } F'(x) = f(x)$$

$$LHS = d\left(\int f(x)dx\right) = d\left(F(x)+C\right) = (F(x)+C)'dx =$$

$$= (F'(x)+0)dx = f(x)dx = RHS.$$

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Theorem 5. If $F \in C^1$ then $\int F'(x)dx = F(x) + C.$

Proof:

It's obvious. ■

Note: Th. 5 may be rewritten as

$$\int du = u + C, \text{ where } u \in C^1.$$

Theorem 6. If f and g are continuous and a is a real constant, then:

1) $\int af(x)dx = a\int f(x)dx$

2) $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx.$

Examples:

$$1)\ \int \frac{\left(1-\sqrt{x}\right)^2}{x^3}dx = \int \frac{1-2\sqrt{x}+x}{x^3}dx = \int \left(\frac{1}{x^3} - \frac{2\sqrt{x}}{x^3} + \frac{x}{x^3}\right)dx =$$

$$= \int x^{-3}dx - 2\int x^{-\frac{5}{2}}dx + \int x^{-2}dx =$$

$$= \frac{x^{-2}}{-2} - 2\cdot\frac{x^{-3/2}}{-3/2} + \frac{x^{-1}}{-1} + C =$$

$$= -\frac{1}{2x^2} + \frac{4}{3x\sqrt{x}} - \frac{1}{x} + C.$$

$$2)\ \int \frac{5-3\sin^2 x}{\cos^2 x}dx = \int \frac{5-3(1-\cos^2 x)}{\cos^2 x}dx =$$

$$= \int \frac{2+3\cos^2 x}{\cos^2 x}dx = \int \left(\frac{2}{\cos^2 x}+3\right)dx =$$

$$= 2\tan x + 3x + C.$$

Theorem 7 (Substitution Rule). If f, g, g' are continuous and F is an antiderivative of f, then

$$\int f[g(x)]g'(x)\,dx = F[g(x)] + C.$$

Proof:

We know that $F'(t) = f(t)$. We have to show that the derivative of RHS is equal to the integrand of LHS. Indeed:

$$\left(F[g(x)]+C\right)' = F'[g(x)]g'(x) + 0 = f[g(x)]g'(x).$$

by the Chain Rule

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How to remember Substitution Rule?

$$\int f[g(x)]g'(x)\,dx = \begin{bmatrix} t = g(x) \\ dt = g'(x)dx \end{bmatrix}$$

t dt

$$= \int f(t)\,dt = F(t) + C = F[g(x)] + C.$$

$g(x)$

Note:

1. Although dx is only the symbol of end of integral, it is convenient to treat dx as differential if we want to apply Substitution Rule.
2. Remember to put original variable at the end of calculations with Substitution Rule.

Examples:

$$1)\ \int \frac{\cos x}{\sin^3 x}dx = \left[\begin{array}{c} t = \sin x \\ dt = \cos x\, dx \end{array} \right] =$$

$$= \int \frac{dt}{t^3} = \int t^{-3} dt = \frac{t^{-2}}{-2} + C = -\frac{1}{2t^2} + C = -\frac{1}{2\sin^2 x} + C$$

$$2)\ \int \frac{dx}{x \ln x} = \left[\begin{array}{c} t = \ln x \\ dt = \frac{1}{x}dx = \frac{dx}{x} \end{array} \right] =$$

$$= \int \frac{dt}{t} = \ln|t| + C = \ln|\ln x| + C.$$

$$3)\ \int \frac{f'(x)}{f(x)}dx = \begin{bmatrix} t = f(x) \\ dt = f'(x)\,dx \end{bmatrix} =$$

$$= \int \frac{dt}{t} = \ln|t| + C = \ln|f(x)| + C.$$

$$4)\ \int \frac{e^x - 1}{e^{2x}}dx = \begin{bmatrix} t = e^x \\ x = \ln t \\ dx = \frac{1}{t}dt \end{bmatrix} = \int \frac{t-1}{t^2} \cdot \frac{1}{t}dt = \int \frac{t-1}{t^3}dt =$$

$$= \int \left(\frac{1}{t^2} - \frac{1}{t^3} \right) dt = -\frac{1}{t} + \frac{1}{2t^2} + C = -\frac{1}{e^x} + \frac{1}{2e^{2x}} + C.$$

Theorem 8 (integration by parts). If $u, v \in C^1$ then

$$\int u\,dv = uv - \int v\,du.$$

Proof:

$$d(uv) = u\,dv + v\,du$$

Property of differential (see Part 6)

$$\int d(uv) = \int u\,dv + \int v\,du$$

$$\int u\,dv = \int d(uv) - \int v\,du$$

it is equal to *uv* (see Th.5)

$$\int u\,dv = uv - \int v\,du$$

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Note: since $du = u'dx$ and $dv = v'dx$, Th. 7 may be written as

$$\int uv'\,dx = uv - \int u'v\,dx$$

or

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Examples:

$$1)\ \int xe^{2x}dx = \begin{bmatrix} u = x & v' = e^{2x} \\ u' = 1 & v = \frac{1}{2}e^{2x} \end{bmatrix} =$$

$$= \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{2}\cdot\frac{1}{2}e^{2x} + C =$$

$$= \frac{1}{4}e^{2x}(2x-1) + C.$$

$$2)\ \int \ln x dx = \int \ln x \cdot 1 dx = \begin{bmatrix} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{bmatrix} =$$

$$= x\ln x - \int 1dx = x\ln x - x + C.$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$3)\ \int e^{2x}\sin x dx = \begin{bmatrix} u = e^{2x} & v' = \sin x \\ u' = 2e^{2x} & v = -\cos x \end{bmatrix} =$$

$$= -e^{2x}\cos x - \int 2e^{2x}(-\cos x)dx =$$

$$= -e^{2x}\cos x + 2\int e^{2x}\cos x dx = \begin{bmatrix} u = e^{2x} & v' = \cos x \\ u' = 2e^{2x} & v = \sin x \end{bmatrix} =$$

$$= -e^{2x}\cos x + 2\left\{e^{2x}\sin x - \int 2e^{2x}\sin x dx\right\} =$$

$$= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x dx$$

$$\int e^{2x} \sin x dx = -e^{2x} \cos x + 2e^{2x} \sin x - 4\int e^{2x} \sin x dx$$

$$5\int e^{2x} \sin x dx = -e^{2x} \cos x + 2e^{2x} \sin x + C_1$$

$$\int e^{2x} \sin x dx = \frac{1}{5}\left(-e^{2x} \cos x + 2e^{2x} \sin x\right) + C$$

$$\int e^{2x} \sin x dx = \frac{e^{2x}}{5}\left(2\sin x - \cos x\right) + C.$$

Integration of rational functions

Partial fractions:

I $\dfrac{A}{x-a}$

III $\dfrac{Ax+B}{ax^2+bx+c}$

II $\dfrac{A}{(x-a)^k}$

IV $\dfrac{Ax+B}{(ax^2+bx+c)^k}$

where $k \in \mathbf{N}, \qquad k \geq 2, \qquad \Delta = b^2 - 4ac < 0$

I $\int \frac{3}{x-5}dx = 3\int \frac{dx}{x-5} = 3\ln|x-5| + C$

II $\int \frac{3}{(x-5)^7}dx = \begin{bmatrix} x-5=t \\ dx=dt \end{bmatrix} = 3\int \frac{dt}{t^7} =$

$3\frac{t^{-6}}{-6} + C = -\frac{1}{2(x-5)^6} + C$

IIIa $\int \frac{1}{x^2-2x+5}dx = \int \frac{dx}{(x-1)^2+4} =$

$$= \frac{1}{4}\int \frac{dx}{\frac{(x-1)^2}{4}+1} = \begin{bmatrix} t = \frac{x-1}{2} \\ dt = \frac{dx}{2} \\ dx = 2dt \end{bmatrix} = \frac{1}{4}\int \frac{2dt}{t^2+1} =$$

$$= \frac{1}{2}\arctan t + C = \frac{1}{2}\arctan\frac{x-1}{2} + C$$

constant times derivative of denominator plus constant

$$\text{IIIb} \quad \int \frac{x+3}{x^2-2x+5}\,dx = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x+5}\,dx =$$

$$= \frac{1}{2}\int \frac{(2x-2)}{x^2-2x+5}\,dx + 4\int \frac{dx}{x^2-2x+5} =$$

$$= \frac{1}{2}\ln|x^2-2x+5| + 2\arctan\frac{x-1}{2} + C$$

IVa $\int \frac{dx}{\left(x^2+1\right)^2} = ?$

$$\arctan x + C_1 = \int \frac{dx}{x^2+1} = \begin{bmatrix} u = \frac{1}{x^2+1} & v' = 1 \\ u' = \frac{-2x}{\left(x^2+1\right)^2} & v = x \end{bmatrix} =$$

$$= \frac{x}{x^2+1} + \int \frac{2x^2}{\left(x^2+1\right)^2} dx = \frac{x}{x^2+1} + 2\int \frac{(x^2+1)-1}{\left(x^2+1\right)^2} dx =$$

$$= \frac{x}{x^2+1} + 2\arctan x - 2\int \frac{dx}{\left(x^2+1\right)^2}$$

$$\int \frac{dx}{\left(x^2+1\right)^2} = \frac{1}{2}\left(\frac{x}{x^2+1} + \arctan x\right) + C$$

IVa $$\int \frac{dx}{\left(x^2+1\right)^3} = ?$$

IVa $$\int \frac{dx}{\left(x^2+1\right)^4} = ?$$

IVa $$\int \frac{dx}{\left(2x^2+6x+9\right)^2} = \frac{1}{4}\int \frac{dx}{\left(x^2+3x+\frac{9}{2}\right)^2} =$$

$$= \frac{1}{4}\int \frac{dx}{\left[\left(x+\frac{3}{2}\right)^2+\frac{9}{4}\right]^2} = \frac{1}{4\cdot\left(\frac{9}{4}\right)^2}\int \frac{dx}{\left[\frac{4\left(x+\frac{3}{2}\right)^2}{9}+1\right]^2} =$$

$$= \left[\begin{array}{c} \frac{2(x+3/2)}{3}=t \\ \frac{2}{3}dx=dt \\ dx=\frac{3}{2}dt \end{array}\right] = \frac{4}{81}\int \frac{\frac{3}{2}dt}{\left[t^2+1\right]^2} = \frac{2}{27}\int \frac{dt}{\left(t^2+1\right)} = \ldots.$$

as in IIIb

$$\text{IVb}\quad \int \frac{3x-1}{\left(2x^2+6x+9\right)^2}\,dx = \int \frac{\frac{3}{4}(4x+6)-\frac{11}{2}}{\left(2x^2+6x+9\right)^2}\,dx =$$

$$= \frac{3}{4}\int \frac{(4x+6)}{\left(2x^2+6x+9\right)^2}\,dx - \frac{11}{2}\int \frac{dx}{\left(2x^2+6x+9\right)^2} = \ldots$$

see IVa

substitute $t = 2x^2+6x+9$

Now we can integrate all partial fractions (sometimes it is very time consuming but it is possible).

Rational function: $f(x) = \dfrac{P(x)}{Q(x)}, \quad P, Q - \text{polynomials}$

$\deg(P) < \deg(Q) \Rightarrow f(x)$ is a proper fraction

$\deg(P) \geq \deg(Q) \Rightarrow f(x)$ is an improper fraction

↓ long division

$$\frac{P(x)}{Q(x)} = W(x) + \frac{R(x)}{Q(x)} \quad \text{where } \deg(R) < \deg(Q)$$

proper fraction

Theorem 9. Any proper fraction can be expressed as a finite sum of partial fractions.

Corollary. We can integrate all rational functions.

How to decompose a proper fraction into partial fractions?

1) factorize the denominator

2) every factor of the form $(x-a)^k$ produces the following sum

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_k}{(x-a)^k}$$

3) every factor of the form $(ax^2+bx+c)^l$ produces the following sum

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_lx+B_l}{(ax^2+bx+c)^l}$$

Example: $$\int \frac{x-3}{x^3-x}\,dx = ?$$

1) $x^3 - x = x(x^2-1) = x(x-1)(x+1)$

2) $$\frac{x-3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \quad \Bigg| \cdot x(x-1)(x+1)$$

$$x-3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$$x-3 = A(x^2-1) + B(x^2+x) + C(x^2-x)$$

$$\begin{aligned} &[x^2]\ \ 0 = A+B+C \\ &[x^1]\ \ 1 = B-C \qquad \Rightarrow [A=3, B=-1, C=-2] \\ &[x^0]\ -3 = -A \end{aligned}$$

3) $$\int \frac{x-3}{x^3-x}dx = \int\left(\frac{3}{x}-\frac{1}{x-1}-\frac{2}{x+1}\right)dx =$$

$$= 3\ln|x| - \ln|x-1| - 2\ln|x+1| + C$$

Example: $\int \frac{2x^3 - x^2 - 3}{x^5 + 3x^3} dx = ?$

1) $x^5 + 3x^3 = x^3(x^2 + 3)$

2) $\frac{2x^3 - x^2 - 3}{x^3(x^2 + 3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + 3}$

$$A = 0,\ B = 0,\ C = -1,\ D = 0,\ E = 2$$

3) $\int \frac{2x^3 - x^2 - 3}{x^5 + 3x^3} dx = \int \left(-\frac{1}{x^3} + \frac{2}{x^2 + 3} \right) dx =$

$$= \frac{1}{2x^2} + \frac{2}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C$$

Example: $$\int \frac{x^3 - 3x^2 + 3x - 6}{(x^2+2)(x^2+3)} dx = ?$$

$$\frac{x^3 - 3x^2 + 3x - 6}{(x^2+2)(x^2+3)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+3}$$

$$A = 1,\ B = 0,\ C = 0,\ D = -3$$

$$\int \frac{x^3 - 3x^2 + 3x - 6}{(x^2+2)(x^2+3)} dx = \int \left(\frac{x}{x^2+2} - \frac{3}{x^2+3} \right) dx =$$

$$= \frac{1}{2} \ln|x^2+2| + \sqrt{3} \arctan \frac{x}{\sqrt{3}} + C$$

Examples:

1) $$\frac{x^2-6}{x(x-5)^3(x^2+2x+3)}=$$

$$=\frac{A}{x}+\frac{B}{x-5}+\frac{C}{(x-5)^2}+\frac{D}{(x-5)^3}+\frac{Ex+F}{x^2+2x+3}$$

2) $$\frac{x+16}{x^2(x+3)(x^2+5)^2}=$$

$$=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+3}+\frac{Dx+E}{x^2+5}+\frac{Fx+G}{(x^2+5)^2}$$

3) $$\frac{x^2+x}{x^2(x+3)(x^2+5)^3}=$$

$$=\frac{A}{x}+\frac{B}{x+3}+\frac{Cx+D}{x^2+5}+\frac{Ex+F}{(x^2+5)^2}+\frac{Gx+H}{(x^2+5)^3}$$