DYNAMICS OF RIGID BODY

MOTION OF MASS IN SPACE

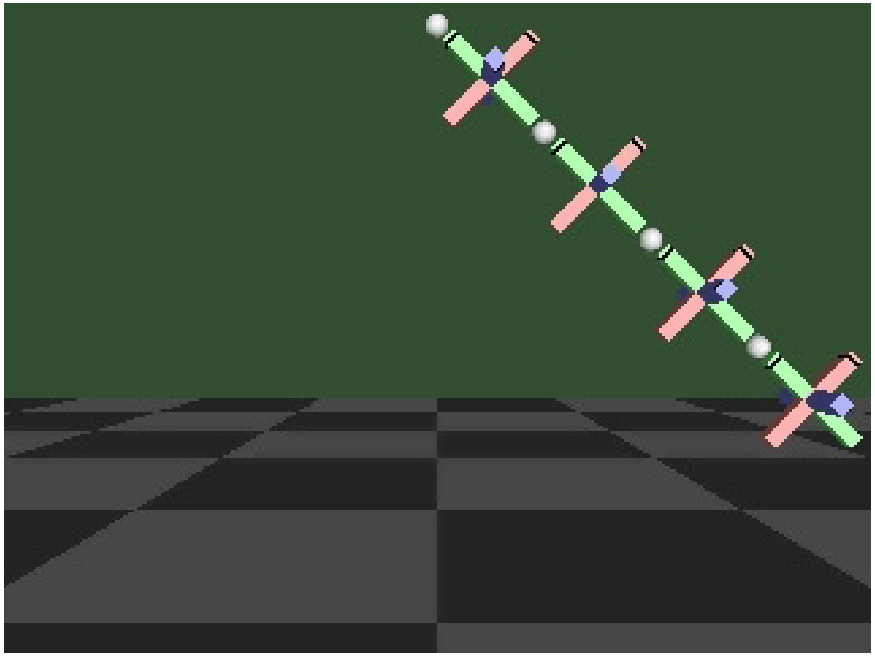
- **forms – domain of kinematics**
- **causes – domain of dynamics**

Main simplification of kinematics and dynamics:
particle treated as material point
for which dimension during a motion with respect to space neglected.

Real body: rigid body

Two main characteristics:

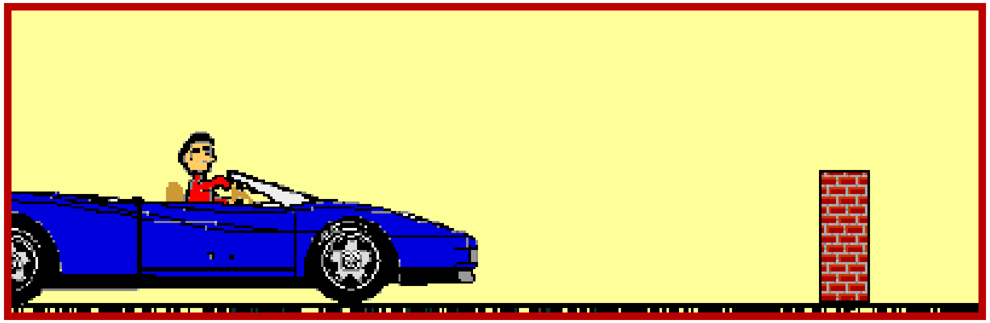
- **finite dimension**
- **stability at action of external forces**



DYNAMICS OF RIGID BODY

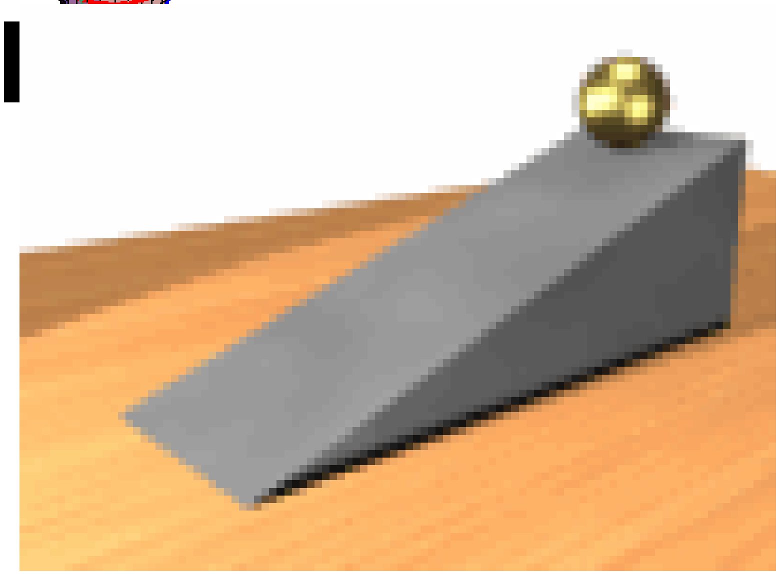
RIGID BODY

System of mass having centre of gravity and containing a large number of material points



Motion of rigid body - combination of two primary simple motions:

- **translation of center of mass (like material point)**
- **rotation of all other material points along fixed rotation axis (motionless) in circles at radius depending on the chosen axis, i.e. A, B**



Primary concept of dynamics of rigid body:
angular momentum

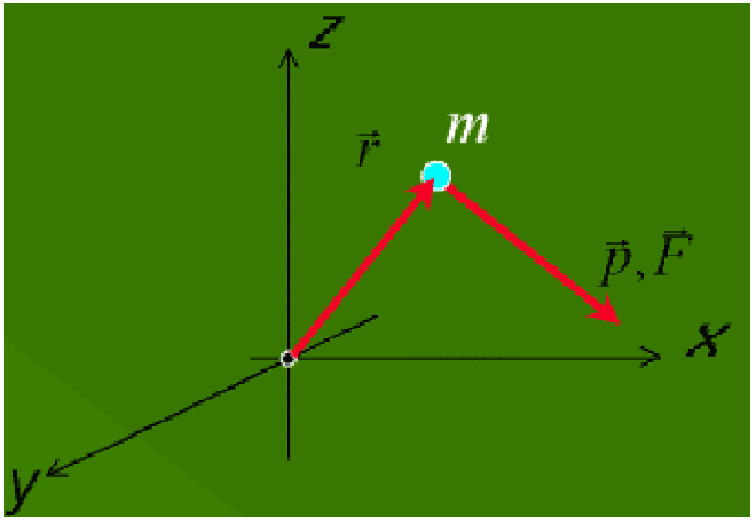
DYNAMICS OF RIGID BODY

Primary concept of dynamics of rigid body – angular momentum

For individual particle of mass m rotating at constant angular velocity ω along fixed axis O

$$L_i = p_i \cdot r_i = m_i \cdot \upsilon_i \cdot r_i = m_i \cdot \omega \cdot r_i \cdot r_i = m_i \cdot \omega \cdot r_i^2$$

Total angular momentum of rigid body rotating along fixed axis at constant angular velocity ω



$$L = \sum_i m_i \cdot \omega \cdot r_i^2 = \omega \sum_i m_i \cdot r_i^2 = \omega \cdot I_A$$

where: $$I_A = \sum_i m_i \cdot r_i^2 \rightarrow \int r^2 dm$$

moment of inertia of rigid body at chosen axis – direct measure of inertia

DYNAMICS OF RIGID BODY

TOTAL ANGULAR MOMENTUM

$$\vec{L} = I_A \cdot \vec{\omega}$$

Motion of rigid body: localized transport of angular momentum (energy)

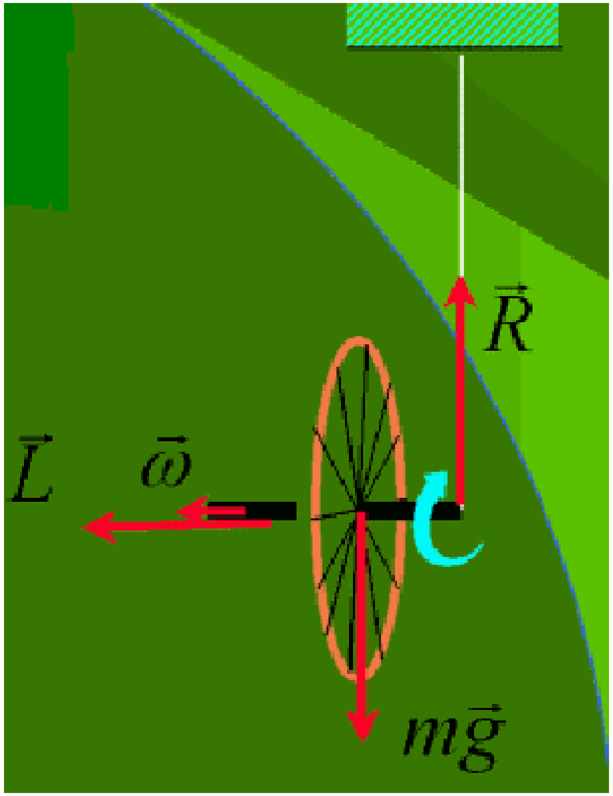
Any change in motion of rigid body - variation of angular momentum as a result of moment of force (torque) *M* acting

$$d\vec{L} = d(I \cdot \vec{\omega}) = I_A \cdot d\vec{\omega} = \vec{M} \cdot dt$$

After transformation

$$\vec{M} = I_A \cdot \frac{d\vec{\omega}}{dt} = I_A \cdot \vec{\varepsilon}$$

- Second dynamics principle for rigid body - Newton's II law

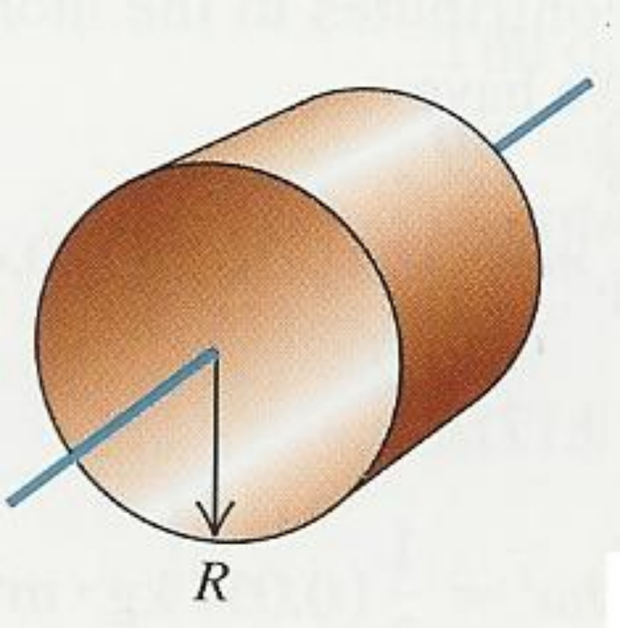


Conclusions:

- **moment of force acting M - cause of rotation**
- **angular acceleration ε - effect (result)**

DYNAMICS OF RIGID BODY

MOMENT OF INERTIA



Basic relation $$I_A = \int r^2 \cdot dm$$

only for high symmetry rigid body for which integration procedure is possible

SIMPLIFIED CASE:
moment of inertia about axis through a centre of body - limited examples

- **solid cylinder about its central axis**

$$I_{SC} = \int_o^R r^2 \cdot dm = \int_o^R r^2 \cdot \rho \cdot dV = \int_o^R r^2 \cdot \rho(2\pi \cdot L \cdot r \cdot dr) = 2\pi \cdot \rho \cdot L \int_o^R r^3 dr$$

after integration and substitution

$$I_{SC} = 2\pi \cdot \rho \cdot L \int_o^R r^3 dr = \frac{1}{2}\pi\rho \cdot L \cdot R^4 = \frac{1}{2}\pi \cdot R^2 \cdot \rho \cdot L \cdot R^2 = \frac{1}{2} m \cdot R^2$$

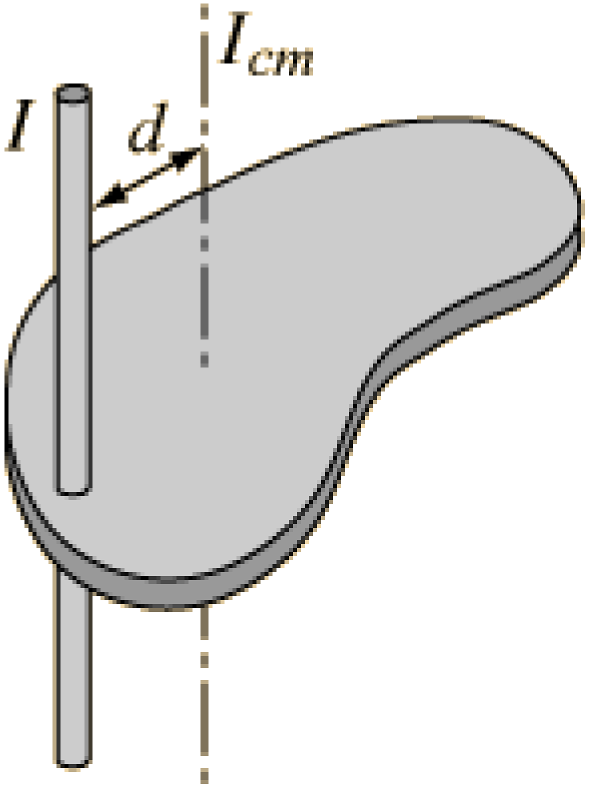
DYNAMICS OF RIGID BODY

MOMENT OF INERTIA

Basic relation $$I_A = \int r^2 \cdot dm$$

only valid for for high symmetry rigid body at axis of rotation through the centre

SPECIFIC CASE:
moment of inertia about axis paralel to an axis through a centre of rigid body



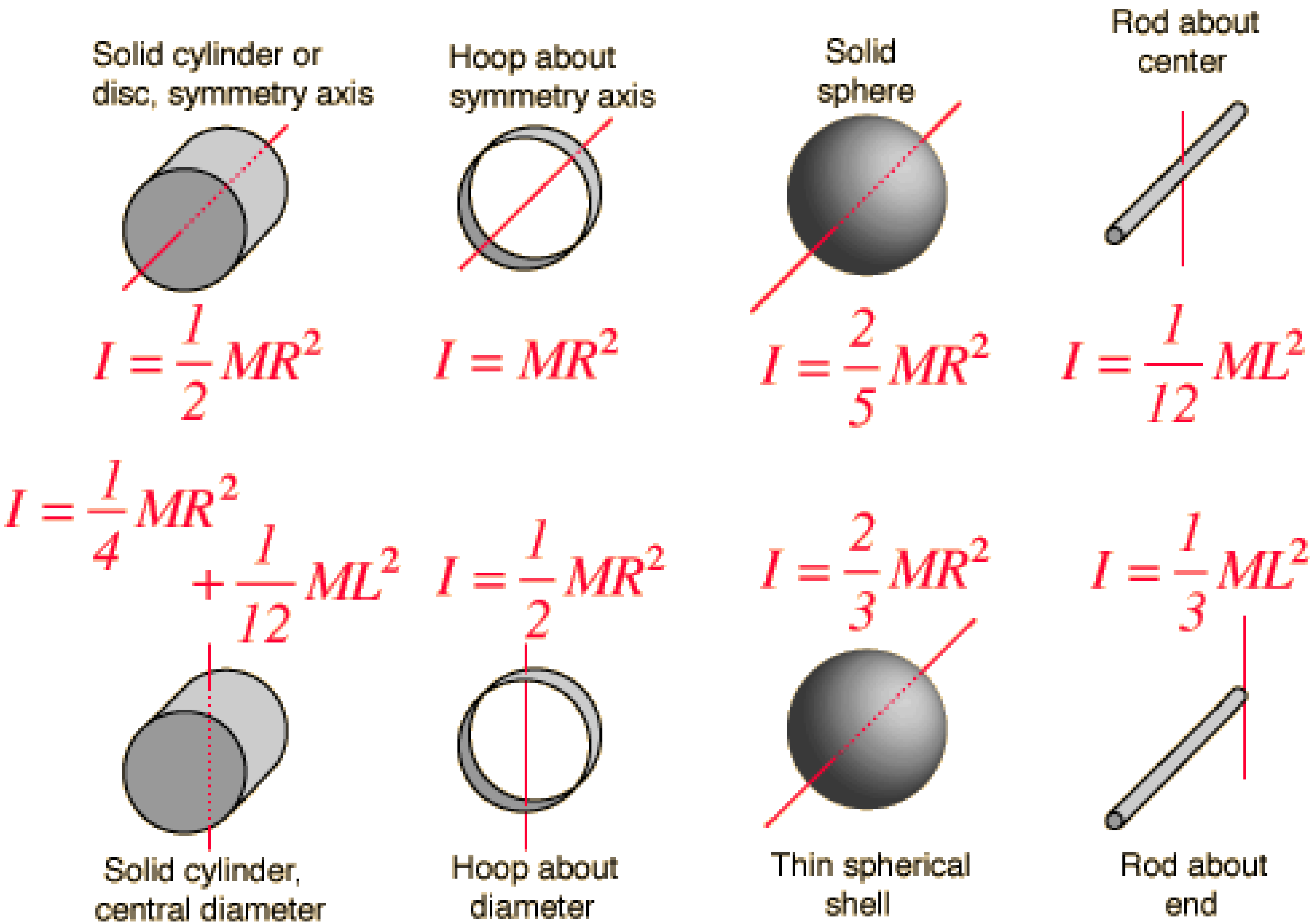
Modified relation:

$$I_{pa} = I_{ma} + m \cdot d^2$$

- Paralel axis theorem - Steiner theorem (rule)

DYNAMICS OF RIGID BODY

MOMENT OF INERTIA OF HIGH SYMMETRY RIGID BODIES



CONSERVATION PRINCIPLES OF RIGID BODY

RIGID BODY

A body having centre of gravity and containg a huge number of material points

Motion of rigid body: combination of two simple motions:

- translation:
change of position of center of gravity under influence of other particle(s) or system via force(s)

- rotation:
change of position of all the points of body under influence of other particle(s) or system via moment of force(s)

Motion of rigid body - localized transport of:
- angular momentum
- energy (two different forms)

determined by respective conservation principles.

CONSERVATION PRINCIPLE OF ANGULAR MOMENTUM

Angular momentu - primary concept of dynamics of rigid body

Under influence of torque (moment of force) applied on rigid body with respect to origin of the reference system

$$\vec{M} = \frac{d\vec{L}}{dt}$$

a variation of angular momentum appears

Only when no torque is acting on body (or net torque is ballanced) angular momentum of rigid body remains constant

$$\vec{L} = I \times \vec{\omega} = const$$

- principle of conservation of angular momentum of rigid body.

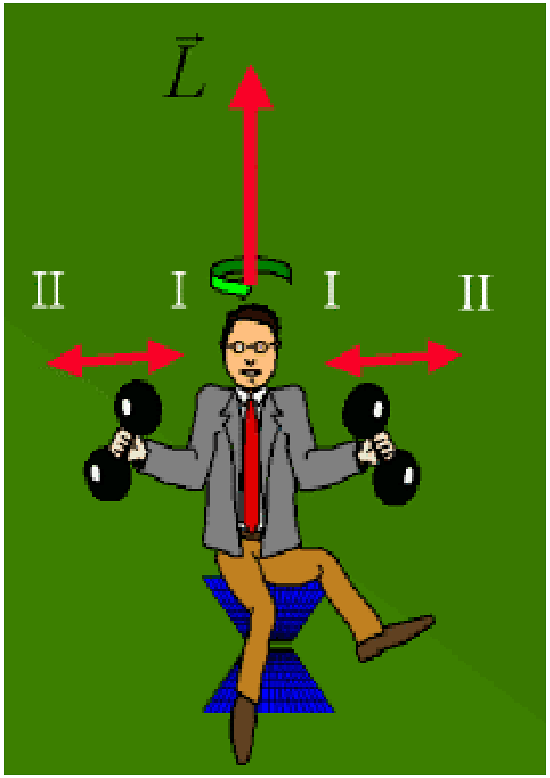
CONSERVATION PRINCIPLE OF ANGULAR MOMENTUM

EXAMPLE 1:

- **Man with handles on rotating chair**

Complex rigid body having one (common) axis of rotation – when system is isolated - a constancy of angular momentum

$$L = I_I \cdot \omega_I = I_{II} \cdot \omega_{II} = const$$



Two possible boundary cases:

- **When moment of inertia with respect to axis of rotation increase (handle position at max. distance) - a decrease of angular velocity of the system**

- **When moment of inertia with respect to axis of rotation (handles position close to man body – axis of rotation) an increasing of angular velocity**

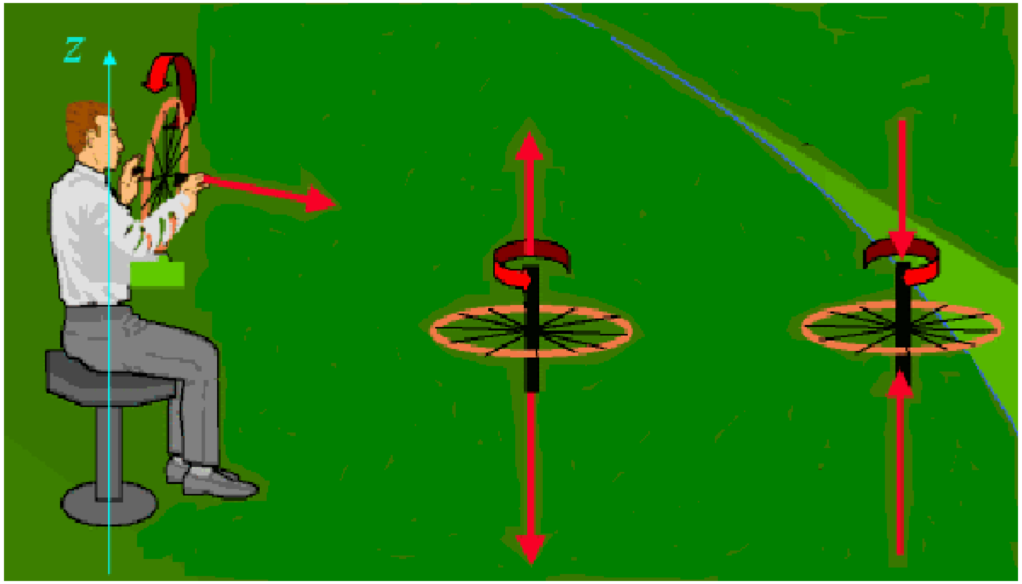
CONSERVATION PRINCIPLE OF ANGULAR MOMENTUM

EXAMPLE 2:

- **Man on rotating chair hanging a rotating wheel**

Complex rigid body having two potential axes of rotation – when system is isolated - a constancy of angular momentum

$$L_{M+W} = 0$$



$$L_t = L_M + L_W = 0$$

Two possible boundary cases:

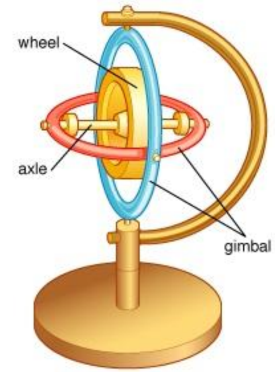
- when wheel rotates perpendicularly to axis of chair – system is in rest

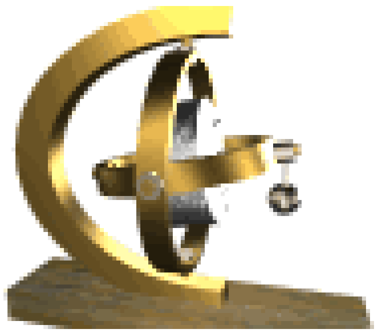
- when axis of wheel rotation is changing with respect to rotation axis of chair - a rotation of chair appears in opposite directions.

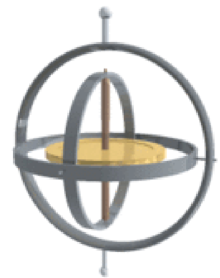
CONSERVATION PRINCIPLE OF ANGULAR MOMENTUM

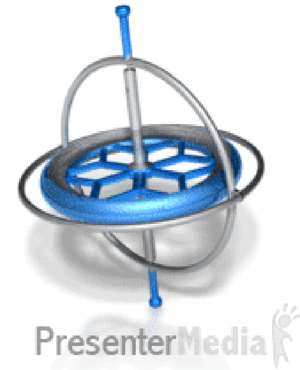
EXAMPLE 3:

• Gyroscope





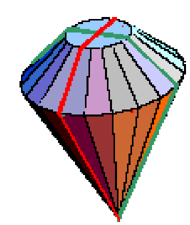




Complex rigid body system – a spinning wheel (disc) in which the axis of rotation is free to assume any orientation by itself. When rotating, an orientation of this axis is unaffected by tilting or mounting rotation - constancy of angular momentum

EXAMPLE 4:

• Rotating top



Only at rotation of top exhibits an angular momentum (vertical) against natural moment of gravitation force - its axis is vertical one

EXAMPLE 5:

• Rotating bicycle wheel on sling

When rotating, an orientation axis is always horizontal (like in bicycle)

CONSERVATION PRINCIPLE OF MECHANICAL ENERGY

FORMS OF MECHANICAL ENERGY OF RIGID BODY

- Kinetic energy

Energy of rigid body during translation and rotation, respectively

$$E_{kt} = \frac{1}{2} m \cdot \upsilon^2 \qquad E_{kr} = \frac{1}{2} I \cdot \omega^2$$

- Potential energy

Energy of center of gravity of rigid body determined by its position in space - mechanical work that have to be done to change a position against gravitation:

$$E_{pcg} = m \cdot g \cdot h$$

For isolated system of rigid body the total mechanical energy remains

$$E_p + E_k = m \cdot g \cdot h + \frac{1}{2} m \cdot \upsilon^2 + \frac{1}{2} I \cdot \omega^2 = const$$

- principle of conservation of mechanical energy of rigid body.

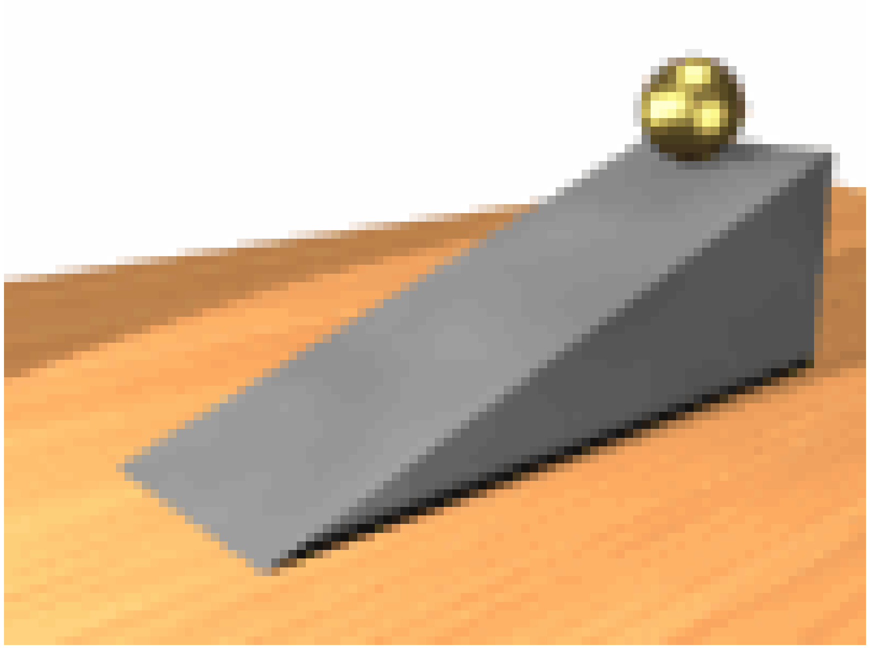
CONSERVATION PRINCIPLE OF MECHANICAL ENERGY

EXAMPLE 1:

- ball on inclined plane

At the top potential energy depends on height *h* of center of gravity

$$E_p = m \cdot g \cdot h$$



Kinetic energy depends on forms of motion and respective velocities

$$E_k = \frac{1}{2} m \cdot \upsilon^2 + \frac{1}{2} I \cdot \omega^2$$

At motion: exchange of energy forms

$$E = E_p + E_k = m \cdot g \cdot h + \frac{1}{2} m \cdot \upsilon^2 + \frac{1}{2} I \cdot \omega^2 = const$$

Typical behaviour for the field of conservative forces - gravitation

CONSERVATION PRINCIPLE OF MECHANICAL ENERGY

EXAMPLE 2:

- **Maxwell wheel (pendulum)**

At the top the potential energy depends on height *h* of center of gravity

$$E_p = m \cdot g \cdot h$$

Kinetic energy depends on forms of motion and respective velocities

$$E_k = \frac{1}{2} m \cdot \upsilon^2 + \frac{1}{2} I \cdot \omega^2$$

At motion: exchange of energy forms

$$E_p + E_k = m \cdot g \cdot h + \frac{1}{2} m \cdot \upsilon^2 + \frac{1}{2} I \cdot \omega^2 = const$$

Typical behaviour for the field of conservative forces - gravitation

DYNAMICS OF RIGID BODY VS MATERIAL POINT

PARAMETER	MATERIAL POINT	RIGID BODY
distance	linear s	angular α
velocity	linear $\upsilon = ds/dt$	angular $\omega = d\alpha/dt$
acceleration	linear $a = d\upsilon/dt$	angular $\varepsilon = d\omega/dt$
cause of motion	force $F = ma$	torque $M = Fr = I\varepsilon$
measure of inertia	mass m	moment of inertia $I = kmr^2$
measure of motion	momentum $p = m\upsilon$	angular momentum $L = I\omega$
forms of energy	kinetics $E_k = ½\, m\upsilon^2$	kinetics: $E_k = ½ m\upsilon^2 + ½\, I\omega^2$
	potential $E_p = mgh$; $E_p = ½\, kx^2$?	