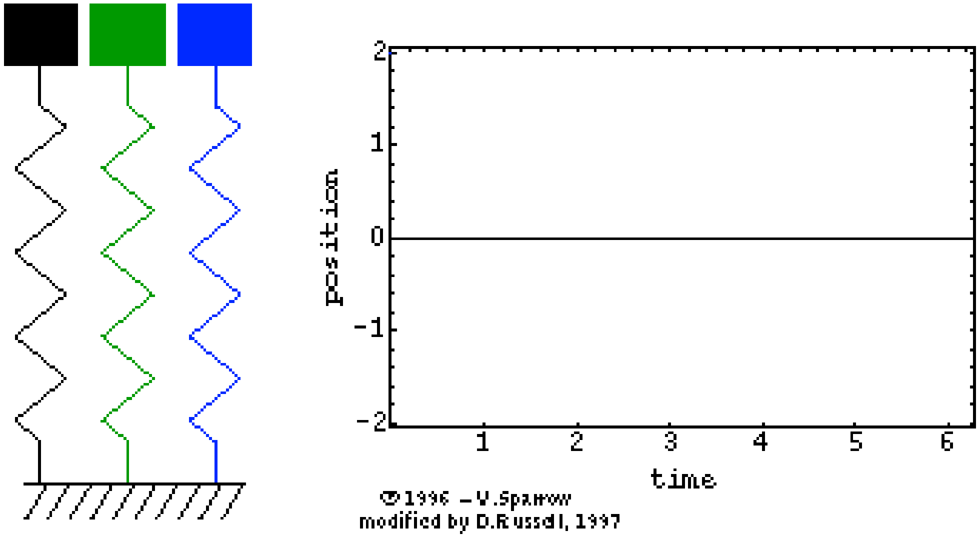
OSCILLATORY MOTIONS - VIBRATIONS

NATURE

A periodical motion (vibration) of particle (material point) with respect to constant equilibrium position



CAUSE

Time dependent force *F(t)* - natural tendency of return of vibrating point to reference (starting) position

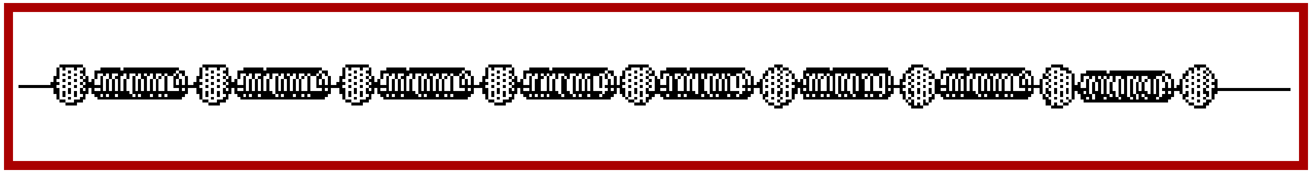
$$x(t) = x(t + T)$$

OSCILLATORY MOTIONS - VIBRATIONS

CLASSIFICATION:

SOURCES:

- **mechanical vibrations: pendulum, quitar string, engine piston, ...**



- **electric oscillations: electric circuits**

FORMS:

- **simple harmonic motion (free vibrations)**
- **damped oscillatory motion (damped vibrations)**
- **damped and then forced (damped and forced vibrations)**
- **complex oscillatory motions (paralell and perpendicular vibrations)**

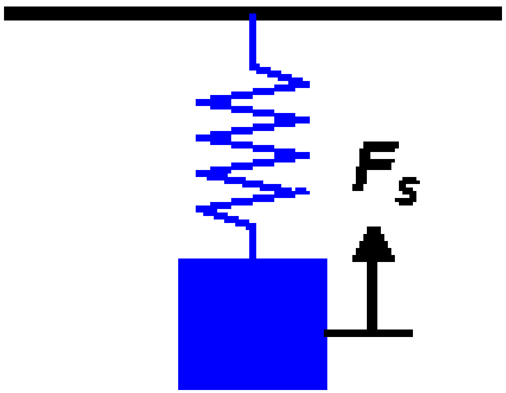
SIMPLE HARMONIC MOTION

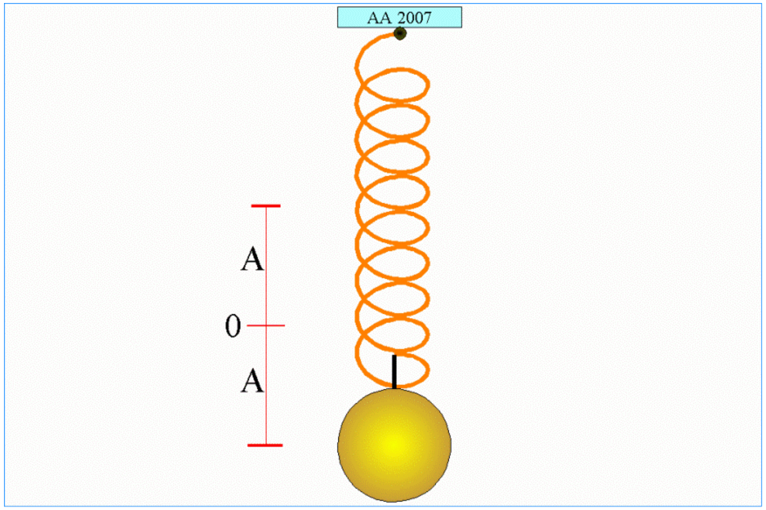
NATURE

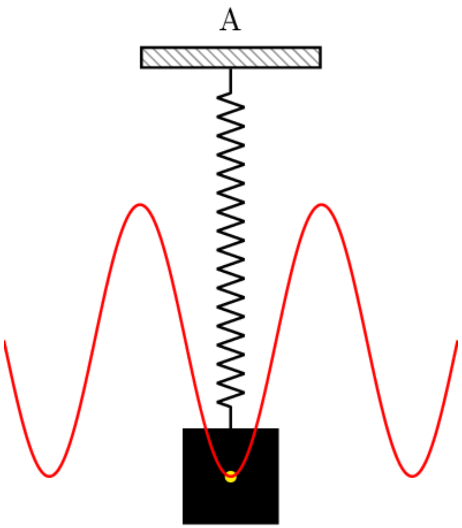
Periodical oscillations of mass (material point) fixed to helical spring with respect to constant equilibrium position – simple harmonic motion

under restoring force $$F_s = -k \cdot x$$

where: *k* - elastic constant according to Hooke's law (unit force)





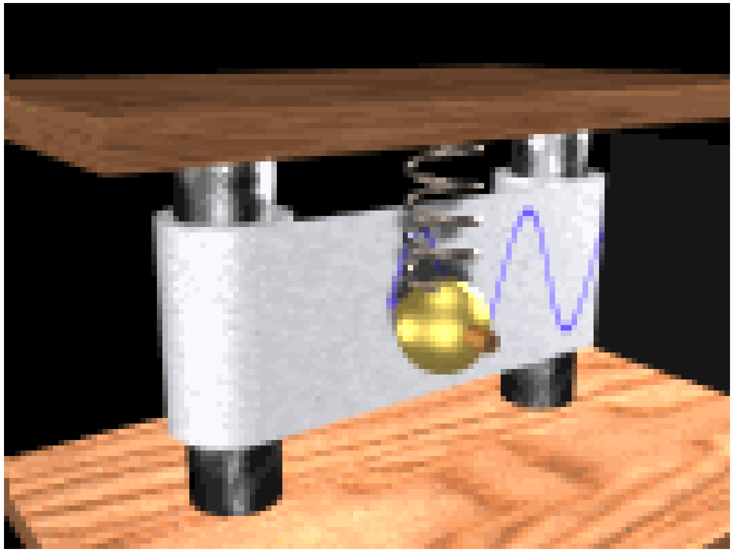


SIMPLE HARMONIC MOTION

NATURE

Periodical oscillations - simple harmonic motion - harmonic oscillator

Time dependent position of oscillating material point along proper axes



$$x = A\cos(\omega \cdot t + \varphi)$$

$$y = A\sin(\omega \cdot t + \varphi)$$

where: A **- amplitude - maximal displacement** x **for which**

$$\sin(\omega \cdot t + \varphi) = 1$$

ω **- angular frequency**

$$\omega = \frac{2\pi}{T} = 2\pi \cdot f$$

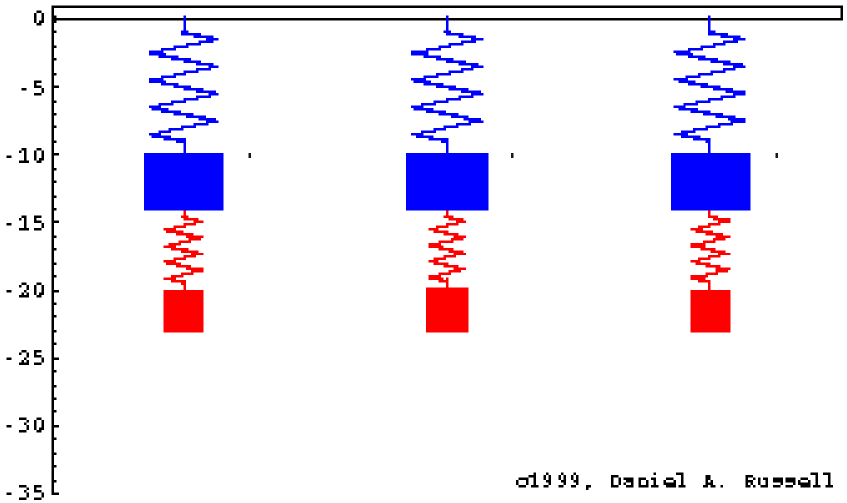
φ **- phase constant (angle)**

KINEMATICS OF SIMPLE HARMONIC MOTION

BASIC KINEMATIC PARAMETERS:

- **displacement** - position with respect to equilibrium

$$x = A\cos(\omega \cdot t + \varphi)$$



- **velocity** - time dep. variation x

$$\upsilon = \frac{dx}{dt} = -A \cdot \omega \cdot \sin(\omega \cdot t + \varphi)$$

- **acceleration** - time dep. variation of

$$a = \frac{d\upsilon}{dt} = \frac{d^2x}{dt^2} = -A \cdot \omega^2 \cdot \cos(\omega \cdot t + \varphi) = -\omega^2 \cdot x$$

- differential equation of harmonic oscillator

DYNAMICS OF SIMPLE HARMONIC MOTION

CAUSE OF VIBRATION

Restoring force - cause of periodical oscillations of mass (material point) with respect to constant equilibrium

$$F = m \cdot a = m\frac{d^2x}{dt^2} = -k \cdot x$$

where: *k* - elastic constant - driving force because for *x=1* $\quad k = |F|$

After transformation

$$m\frac{d^2x}{dt^2} + k \cdot x = 0$$

- differential equation of simple harmonic (free) vibrations
having simplest solution function (displacement)

$$x = A\cos(\omega_o \cdot t + \varphi)$$

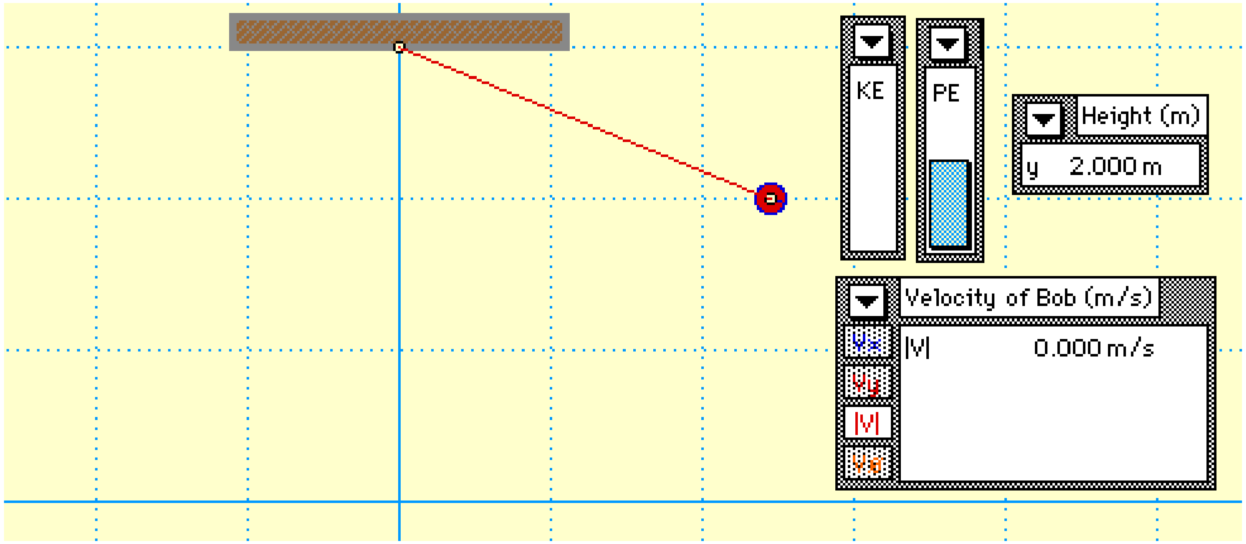
at free vibration frequency: $\omega_o = \sqrt{\frac{k}{m}}$ **and/or period** $T_o = 2\pi\sqrt{\frac{m}{k}}$

DYNAMICS OF SIMPLE HARMONIC MOTION

EXAMPLE:

- simple (mathematical) pendulum

Periodical vibrations (oscillations) of mass *m* on lightweight string of length *l* along circle with respect to equilibrium position



Two boundary cases:

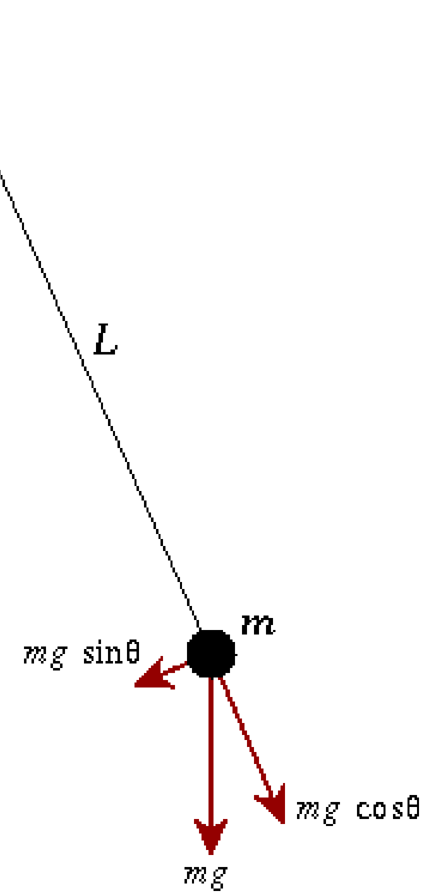
- **at rest:**
  string hangs vertically

- **after deflection**
  restoring force which cause of periodical oscillations of mass (material point) on lightweight with respect to constant equilibrium
  - gravitational force (?)

DYNAMICS OF SIMPLE HARMONIC MOTION

- simple (mathematical) pendulum

Restoring force of simple pendulum - tangential component of gravitational force



$$F_s = m \cdot g_\perp = -m \cdot g \cdot \sin\varphi$$

Only for small angle up to 7°

$$\sin\varphi \approx tg\varphi = \frac{s}{L}$$

thus acceleration of simple pendulum

$$a_s = \frac{d^2 s}{dt^2} = -g \cdot \frac{s}{L} = -\omega^2 \cdot s$$

- differential equation of simple pendulum

oscillation frequency: $\omega_o = \sqrt{\frac{k}{m}}$ and/or period $T_o = 2\pi\sqrt{\frac{m}{k}}$

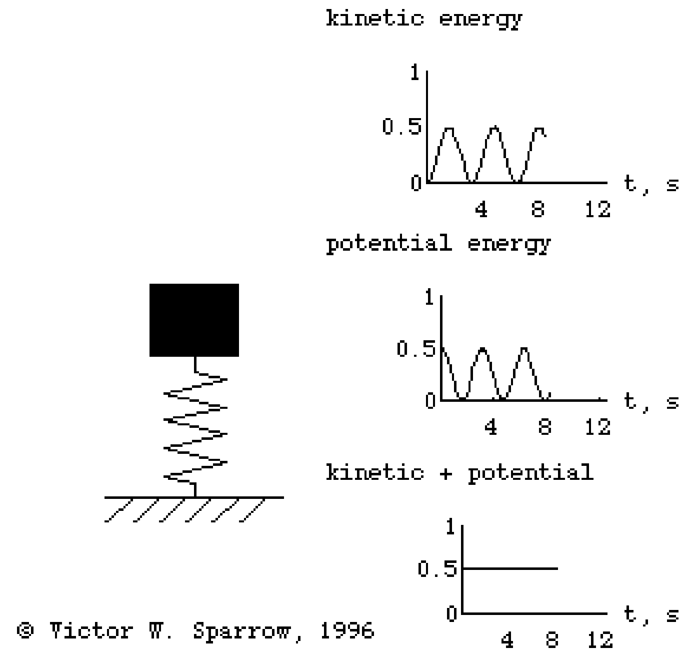
DYNAMICS OF SIMPLE HARMONIC MOTION

ENERGY

Two forms of mechanical energy of oscillating mass (material point): potential and kinetic energy

For harmonic oscillator - displacement of material point on spring from equilibrium position

Mechanical work done by external force for displacement *x* against elastic force of spring - potential energy of spring



$$W = \int_o^x F(x)dx = -k\int_o^x xdx = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \varphi) = E_p$$

transforms on kinetic energy causes return to equilibrium state

$$E_k = \frac{1}{2}m\upsilon^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}mA^2\sin^2(\omega t + \varphi)$$

DYNAMICS OF SIMPLE HARMONIC MOTION

ENERGY

Total mechanical energy of harmonic oscillator

$$E = E_p + E_k = \frac{1}{2}kx^2 + \frac{1}{2}m\upsilon^2 = \frac{1}{2}kx^2 + \frac{1}{2}m(\frac{dx}{dt})^2$$

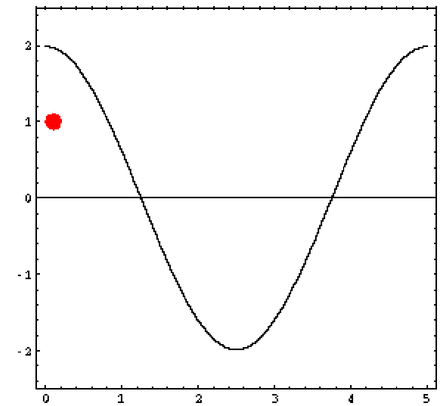
After substitution

$$E = \frac{1}{2}k \cdot A^2 \cos^2(\omega t + \varphi) + \frac{1}{2}m \cdot \omega^2 A^2 \sin^2(\omega t + \varphi)$$

Because $k = m\omega^2$

$$E = \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \varphi) + \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \varphi) = \frac{1}{2}m\omega^2 A^2$$

For isolated simple harmonic motion exchange of potential energy on kinetic energy, and inversely - total mechanical energy remains constant - conservation principle of mechanical energy !



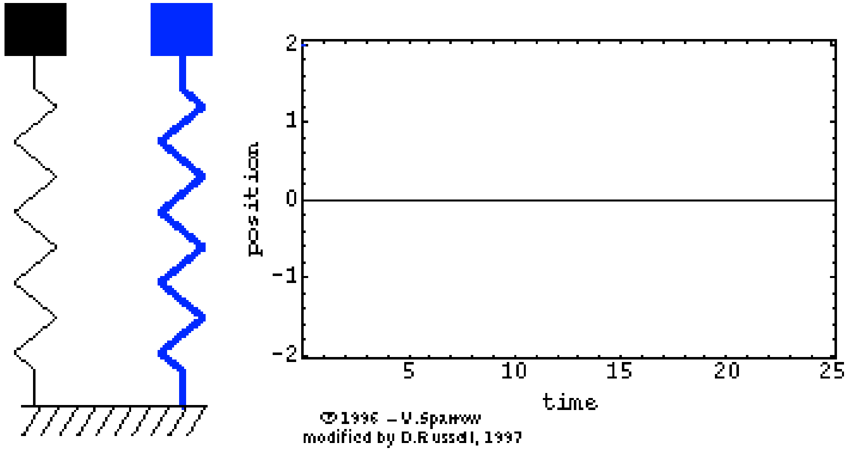
DAMPED OSCILLATORY MOTION

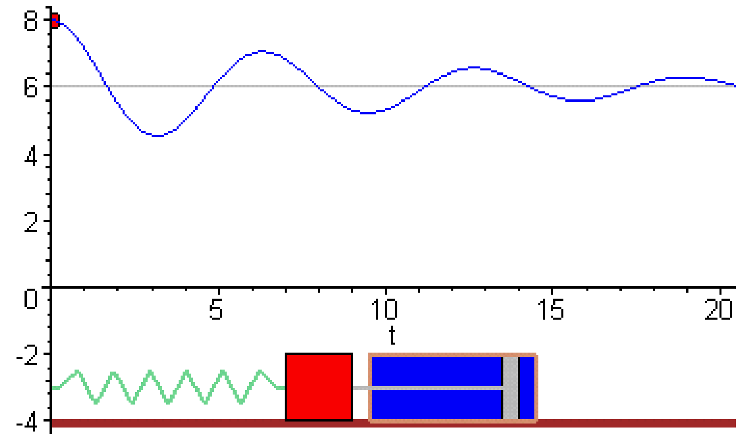
NATURE

Simple harmonic oscillations under influence of elastic force

$$F = m \cdot a = m\frac{d^2x}{dt^2} = -k \cdot x$$

can be damped by force related to medium resistance (internal friction)





- decay of oscillatory motion
- damping force always directed against simple harmonic oscillations

DAMPED OSCILLATORY MOTION

Only for small velocity υ damping force

$$F_d = -b \cdot \upsilon = -b \cdot \frac{dx}{dt}$$

where: b - coefficient of medium resistance (internal friction)

According to II dynamics principle a net force of damped oscillations

$$m\frac{dx^2}{dt^2} + k \cdot x + b \cdot \upsilon = 0$$

- differential equation of damped oscillatory motion

having simplest solution function (displacement) at conditions:

- it reduces to simple oscillations without damping, when $b \to 0$
- it exhibits amplitude reducing in time with damping, when $b \neq 0$

  in form

$$x = e^{-\frac{b}{2m}t} \cdot A_o \cdot cos(\omega \cdot t + \varphi) = e^{-\beta t} \cdot A_o \cdot cos(\omega \cdot t + \varphi)$$

where: $\omega = \sqrt{\omega_o^2 - b^2 / 4m^2}$ - angular frequency of damped oscillations

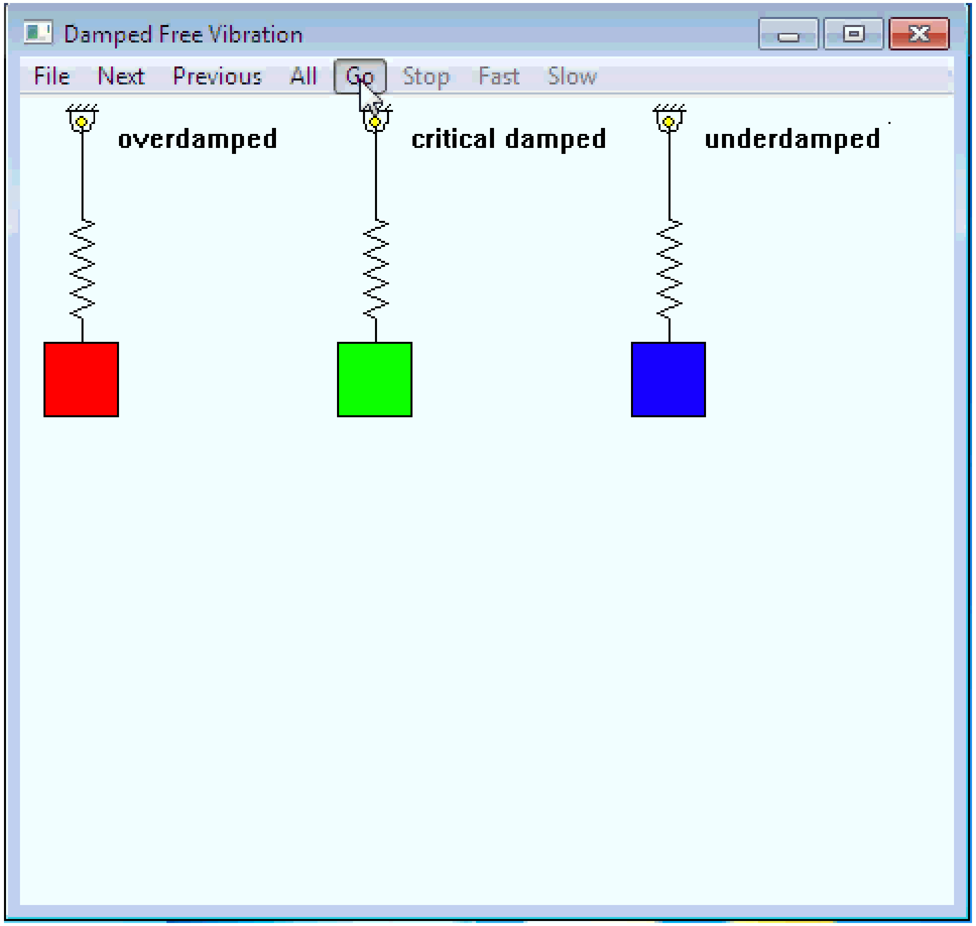
$\beta = \frac{b}{2m}$ - damping coefficient (vibrations decay c.)

DAMPED OSCILLATORY MOTION

Displacement *x(t)* in damped oscillations depends on absolute value of

$$\omega = \sqrt{\omega_o^2 - b^2 / 4m^2}$$

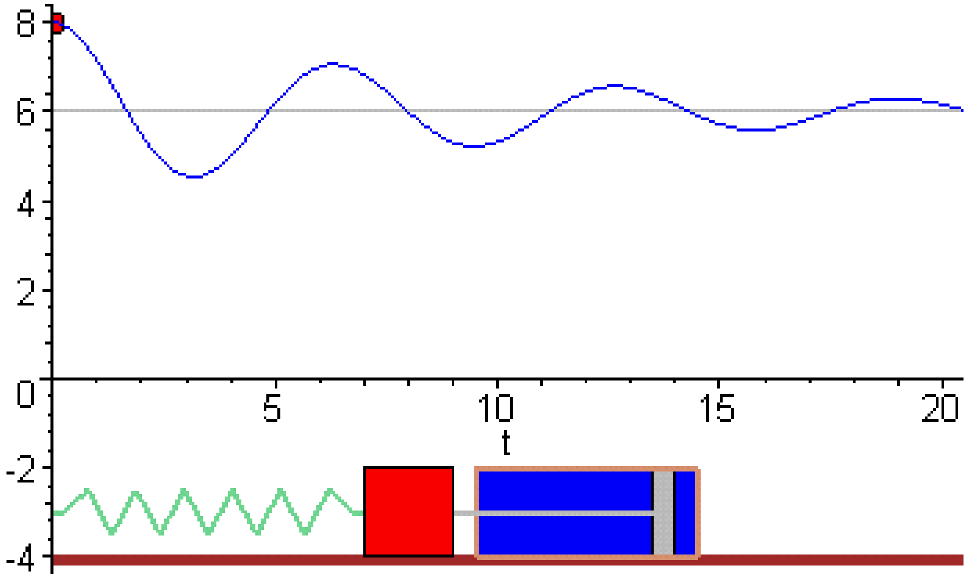
3 boundary conditions:



DAMPED OSCILLATORY MOTION

- weak damping (ω > 0) (underdamping)

$$\frac{k}{m} \succ \frac{b^2}{4m^2}$$



Slow returning of oscillating point to equilibrium state at

- angular frequency $\omega = \sqrt{\omega_o^2 - b^2 / 4m^2} = \sqrt{\omega_o^2 - \beta^2} \prec \omega_o$

- amplitude $A = A_o \cdot e^{-\frac{b}{2m}t} = A_o \cdot e^{-\beta t}$

Final effect: with additional simplification - almost periodic oscillations

DAMPED OSCILLATORY MOTION

For weak damping after proper time corresponding to period T amplitude of damped oscillatory motion

$$A_1 = A_o \cdot e^{-\beta t}$$

After time related to period

$$t = n \cdot T$$

it reaches a form

$$A_n = A_o \cdot e^{-n \cdot \beta \cdot T} = A_o \cdot e^{-n \cdot \Lambda}$$

where:

$$\Lambda = \beta \cdot T$$

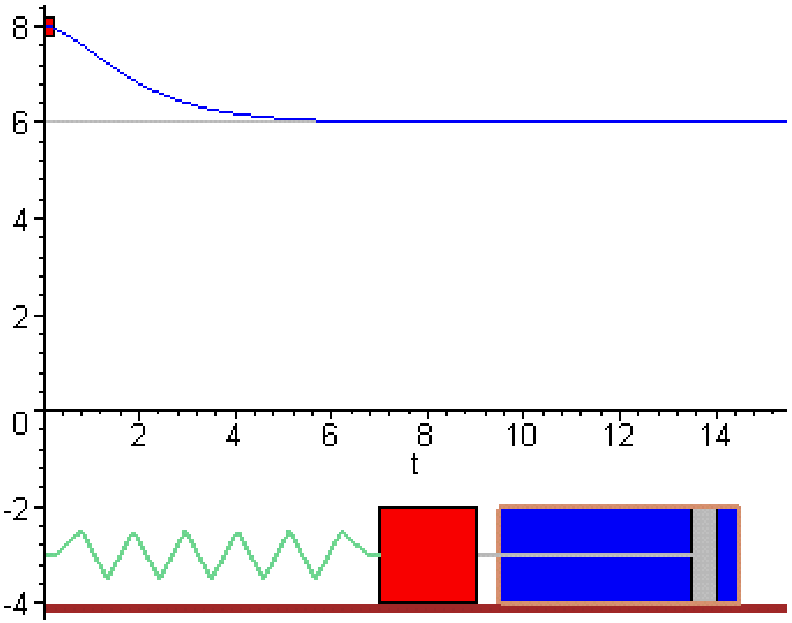
or **- logarithmic decrement of damping**

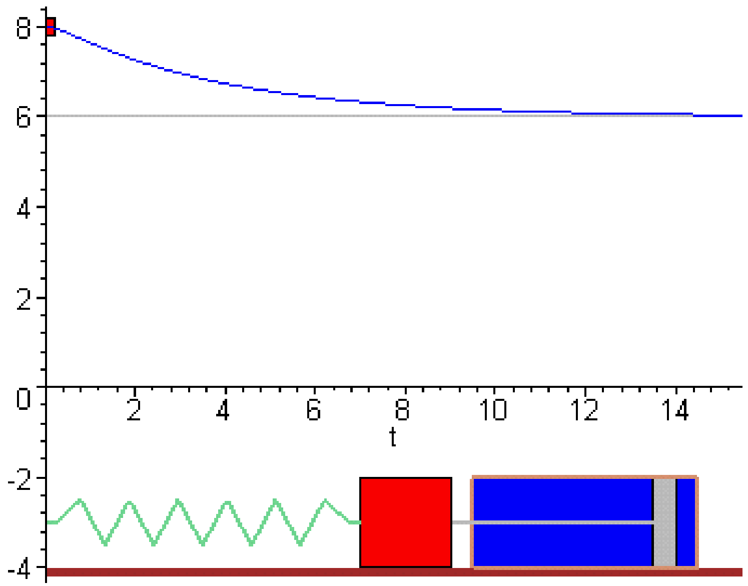
$$\Lambda = \ln \frac{A_n}{A_{n+1}}$$

DAMPED OSCILLATORY MOTION

- **Strong or critical damping (ω = 0)**

$$\frac{k}{m} = \frac{b^2}{4m^2}$$





Final effect:

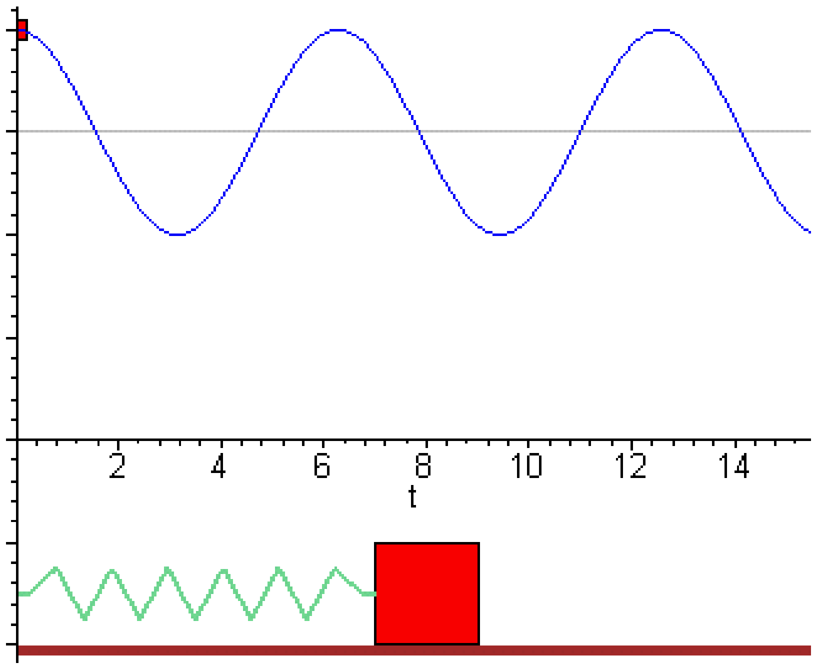
returns of oscillating point to equlibrium without its crossing (line) – aperiodic function of decay

$$x = A_o e^{-\beta t}(1 + \beta \cdot t)$$

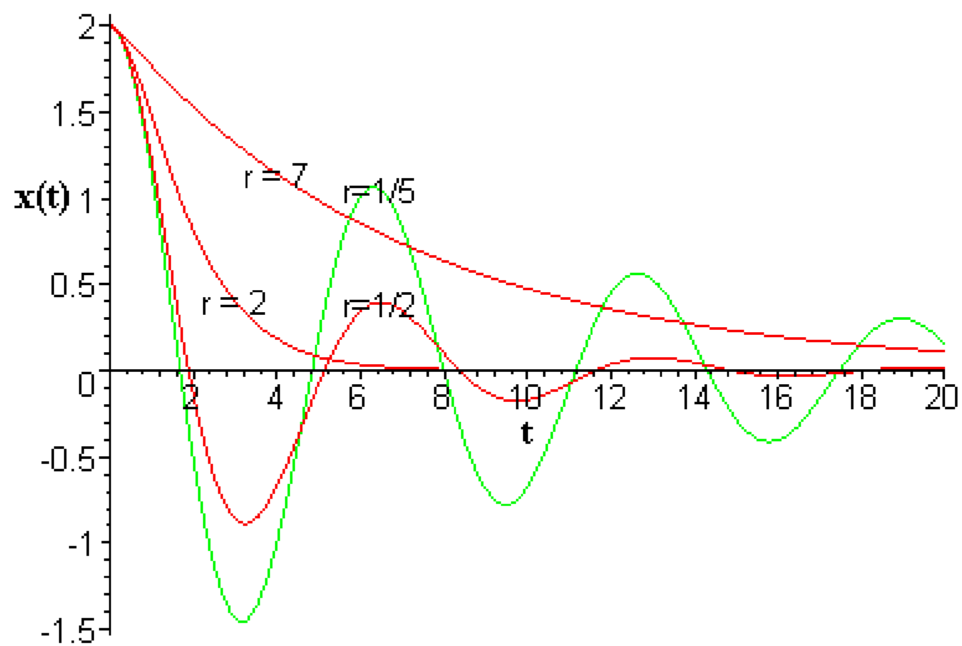
DAMPED OSCILLATORY MOTION

Direct comparison of undamped and damped vibration of different damping mechanism

undamped

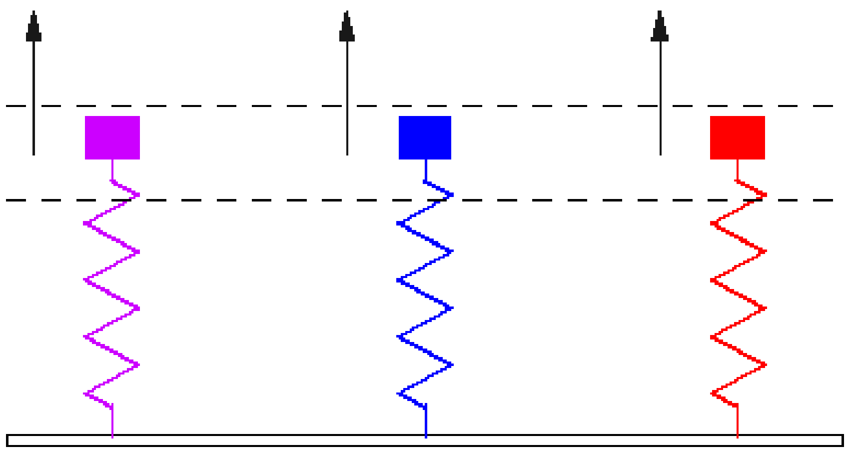


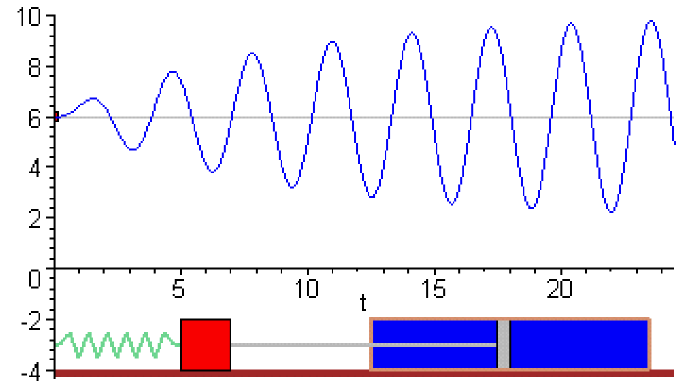
damped for various β=r

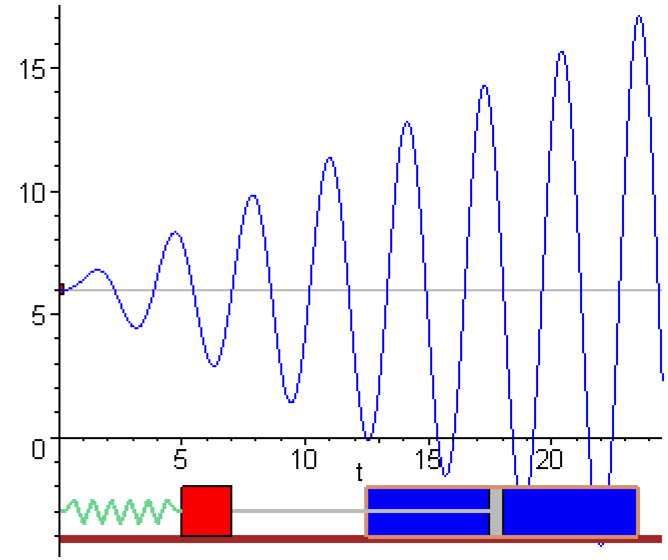


DAMPED AND FORCED OSCILLATORY MOTION

Damped oscillatory motion after decay as a result of damping







can be restored to previous harmonic oscillations by external input force

DAMPED AND FORCED OSCILLATORY MOTION

According to II Newton's law a net force of damped and then forced oscillations

$$F_f = F_o \cdot cos(\Omega \cdot t)$$

Thus

$$m\frac{dx^2}{dt^2} + b\frac{dx}{dt} + k \cdot x = F_o cos(\Omega \cdot t)$$

or

$$\frac{dx^2}{dt^2} + 2\beta\frac{dx}{dt} + \omega_o^2 \cdot x = B \cdot cos(\Omega \cdot t)$$

- differential equation of damped oscillatory motion
having solution (displacement) in form of periodic function

$$x = A \cdot cos(\Omega \cdot t - \Phi)$$

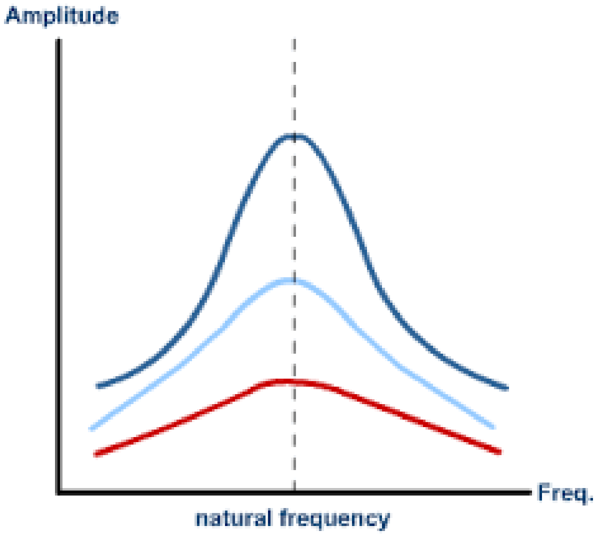
where:

$$A = \frac{B}{\sqrt{(\omega_o^2 - \Omega^2) + 4\beta^2\Omega^2}}$$ **- amplitude**

$$tg\Phi = \frac{2\beta \cdot \Omega}{\omega_o^2 - \Omega^2}$$ **- phase**

DAMPED AND FORCED OSCILLATORY MOTION

Special case: absence of damping $\beta = 0$

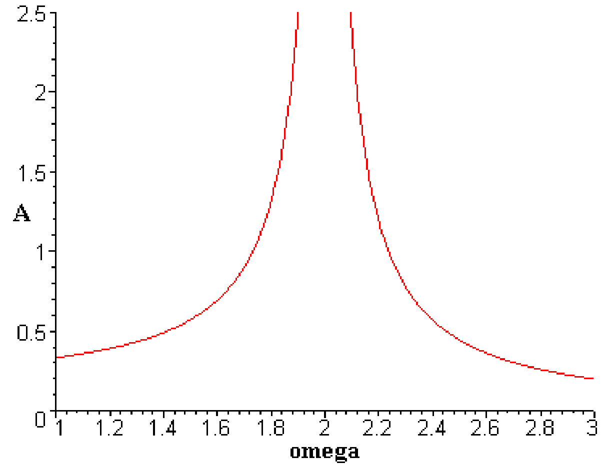


Amplitude $$A = \frac{F_o}{\left| m(\omega_o^2 - \Omega^2) \right|}$$

When $\omega_o \Rightarrow \Omega$

Amplitude $A \Rightarrow \infty$

General case: with damping $\beta \neq 0$



Amplitude:

$$A = \frac{B}{\sqrt{(\omega_o^2 - \Omega^2) + 4\beta^2\Omega^2}}$$

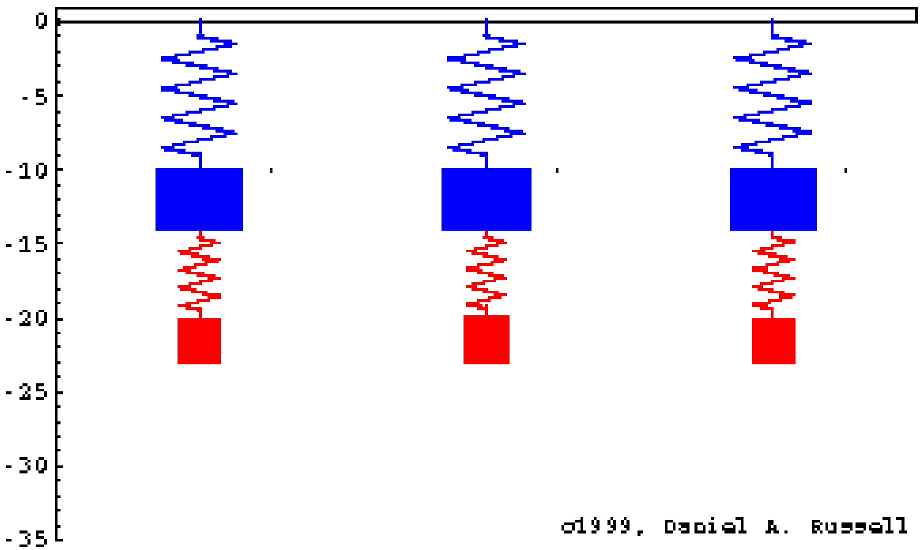
At particular ω and weak damping (small β) amplitude of forced oscillations reaches a maximum - effect of resonance

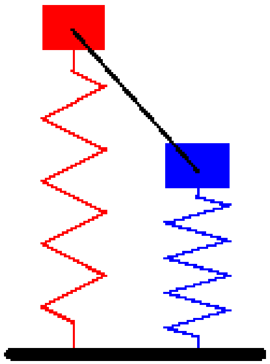
$$A_r = \frac{B}{2\beta\omega}$$

COMPLEX OSCILLATORY MOTION

NATURE

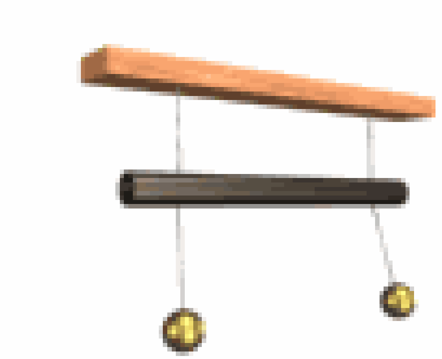
Real oscillations: superposition of different component oscillations





Only possible description - every oscillation treated as superposition of linear harmonic oscillations

$$x_n = A_n \cdot cos(\omega_n t + \varphi_n)$$



Two boundary cases:

- **parallel oscillations**
- **perpendicular oscllations**

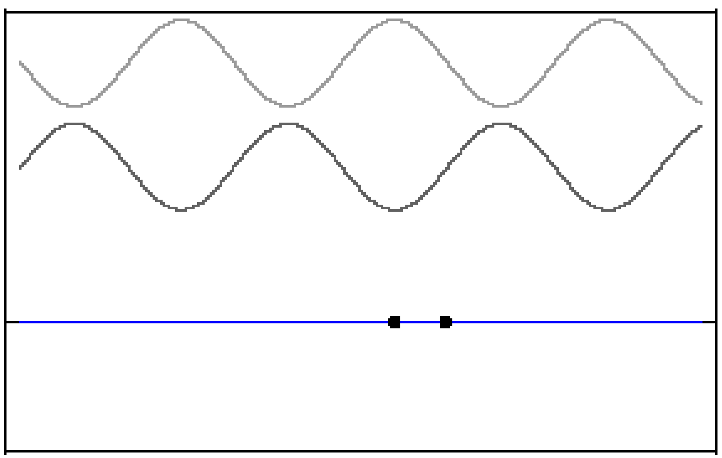
PARALLEL OSCILLATORY MOTION

Superposition of linear harmonic oscillations in one direction – 3 boundary cases:

- at identical angular frequency: $\omega_1 = \omega_2 = \omega = const$

Net displacement

$$x = A \cdot cos(\omega t + \varphi)$$



When

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1 \cdot A_2 \cdot cos(\varphi_2 - \varphi_1)}$$

Net phase

$$tg\varphi = \frac{A_1 \cdot sin\,\varphi_1 + A_2 \cdot sin\,\varphi_2}{A_1 \cdot cos\,\varphi_1 + A_2 \cdot cos\,\varphi_2}$$

Two boundary cases:

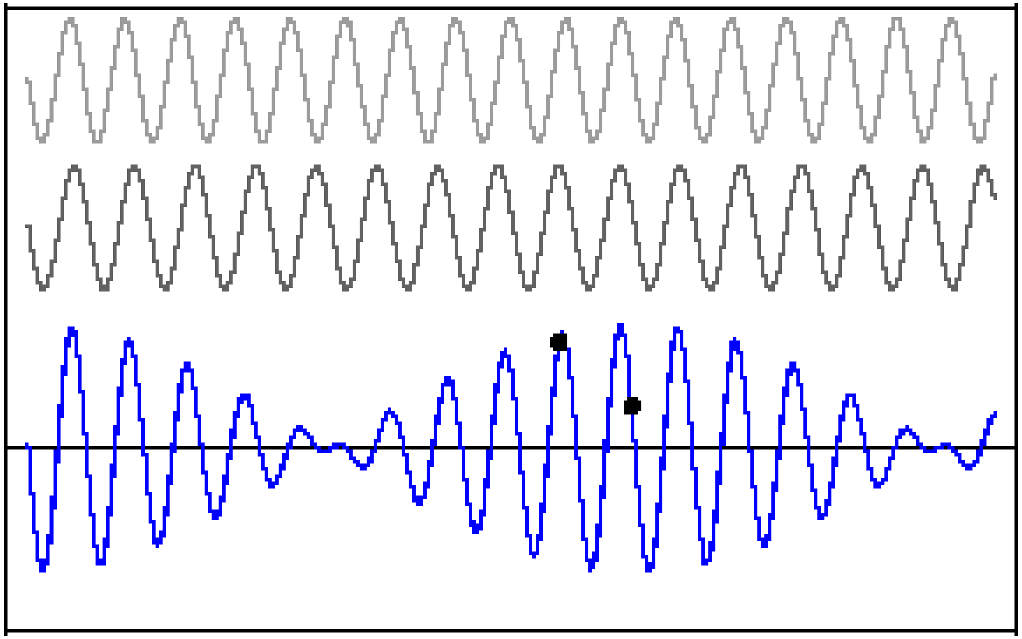
- **phase coincidence - amplitude summation**
- **phase non-coincidence - amplitude substraction**

PARALLEL OSCILLATORY MOTION

- at close angular frequency: $\omega \pm \Delta\omega$

Net displacement

$$x = 2A \cdot \cos\omega \cdot t \cdot \cos\Delta\omega \cdot t$$



Specific case when

$$\Delta\omega \prec 16Hz$$

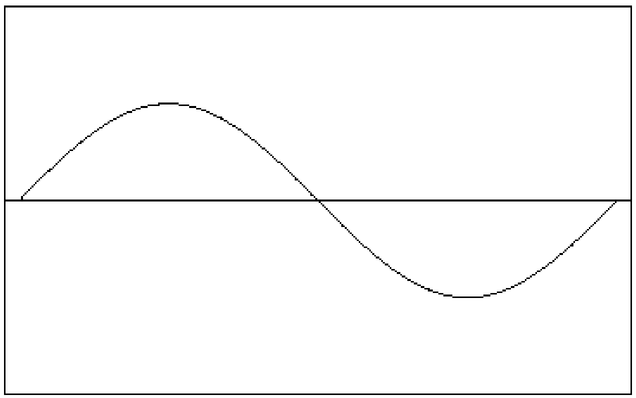
i.e. in the audible region

- ***effect of beat***

PARALLEL OSCILLATORY MOTION

- at different angular frequency: anharmonic oscillations

Only possible solution for arythmetic sequence: $\omega, 2\omega, 3\omega, .. n\omega$



Net displacement

$$x = \sum_{n=1}^{\infty} A_n \cos(n\omega t + \varphi)$$

Component oscillations: first, second, third, ... harmonics

Important inverse problem:

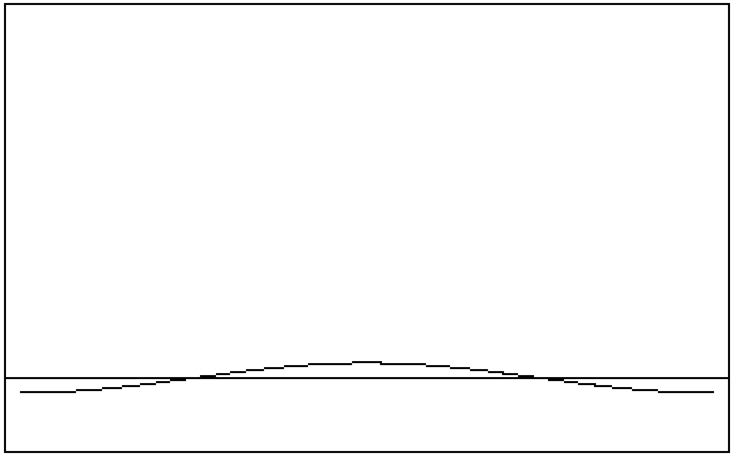
decomposition of arbitrary periodic oscillations for single harmonic oscillations of different amplitudes - Fourier analysis

PARALLEL OSCILLATORY MOTION

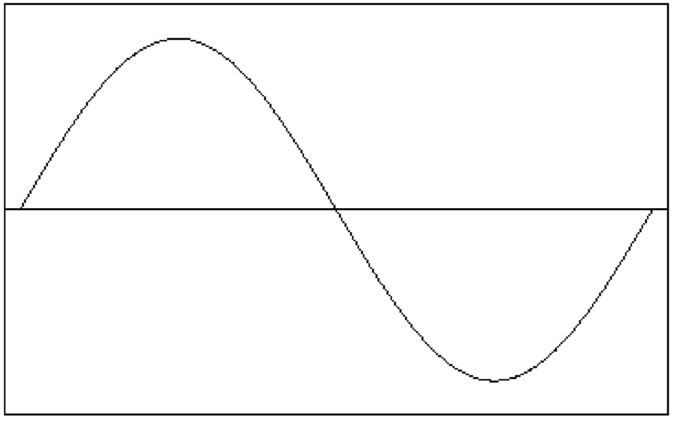
FOURIER ANALYSIS

Two most common examples:

- **pulse type oscillations**



- **square type oscillations**



PARALLEL OSCILLATORY MOTION

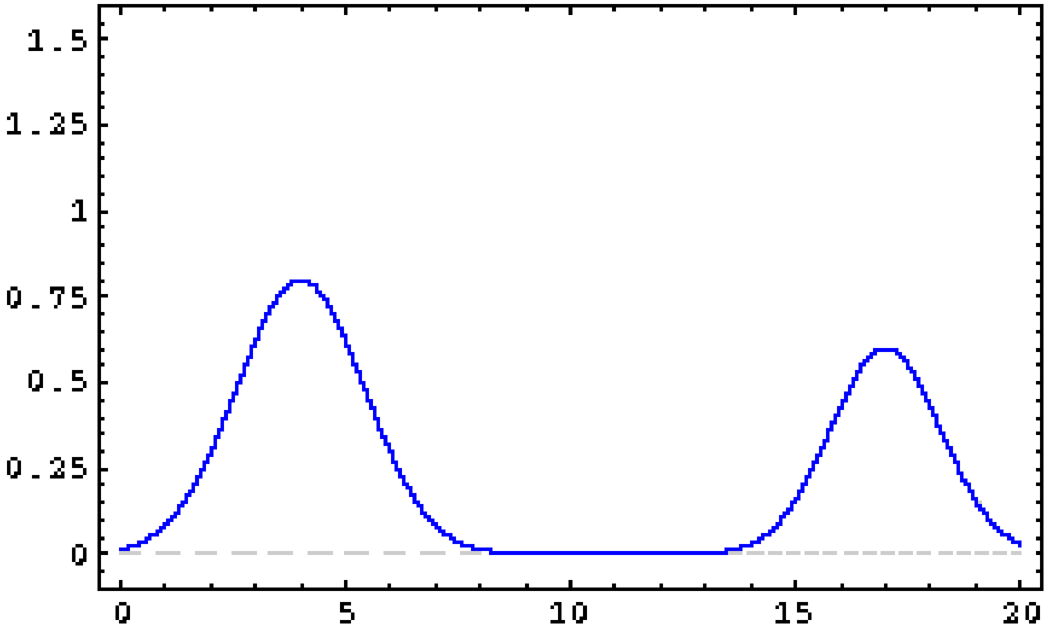
SUPERPOSITION

Simplified case: two pulses

- different amplitude

- different frequency

- oposite direction



Two possible effects:

- convolution of two component pulses (oscillations) to complex form
- deconvolution of complex form into two component pulses (oscillations)

PERPENDICULAR OSCILLATORY MOTION

Superposition of at least two linear harmonic perpendicular oscillations along two perpendicular *x* and *y* axes

Net oscillations - curve at xy plane - Lissajous figures (curves)

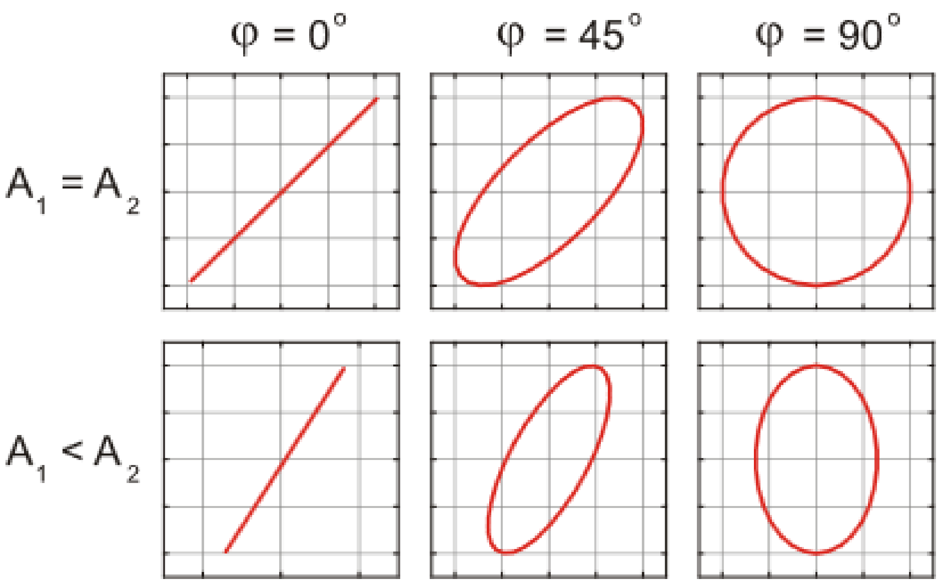
Two boundary cases:

- for identical angular frequency:

Superposition of two components

$$x = A_1 \cdot \cos(\omega t + \varphi_1)$$

$$y = A_2 \cdot \cos(\omega t + \varphi_2)$$



For net oscillations at $\Delta\varphi = \pi / 2$

elipse on *xy* plane $\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$

Specific case: for the same amplitude a net ocillation - a circle

PERPENDICULAR OSCILLATORY MOTION

- **for different angular frequencies:**

superposition of two components

$$x = A_1 \cdot cos(n_1 \cdot \omega t + \varphi_1) \qquad y = A_2 \cdot cos(n_2 \cdot \omega t + \varphi_2)$$

Net oscillations: Lissajous figures (curves)

Most popular case:
electric oscillations

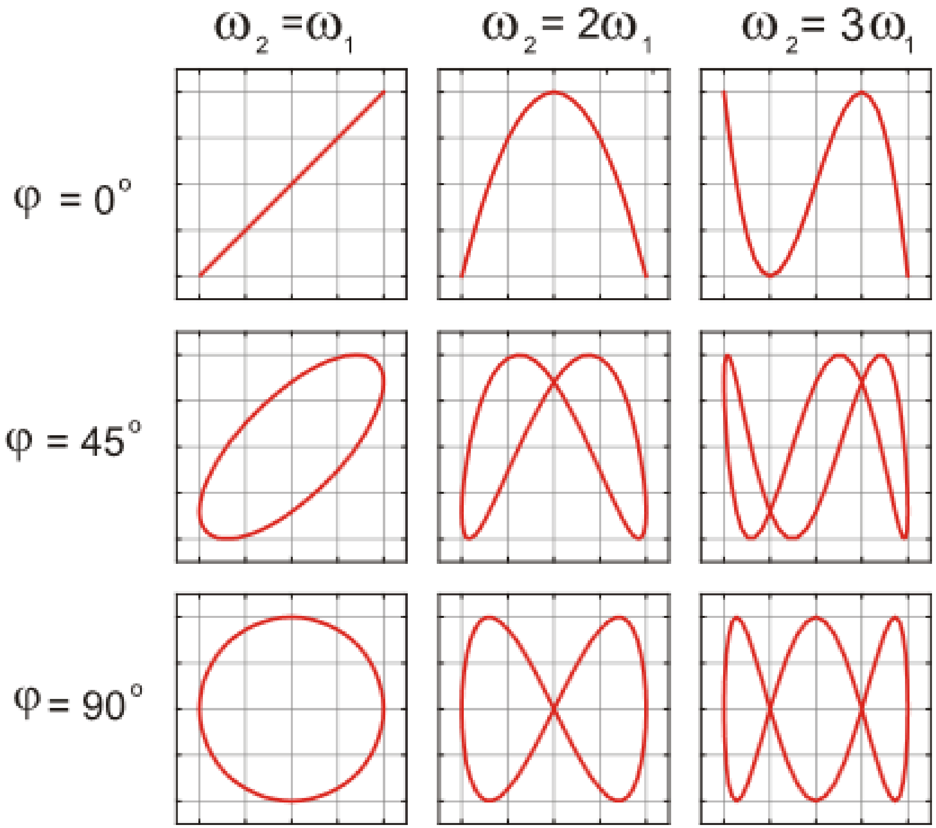
Final shape depends on:

- **angular frequency ratio**

$$\omega_x : \omega_y$$

- **phase shift**

$$\Delta\varphi = \varphi_2 - \varphi_1$$



PERPENDICULAR OSCILLATORY MOTION

- **at different angular frequencies:**
  superposition of two components

Lissajous figures at different angular frequency ratio $\omega_x : \omega_y$